MAXIMAL EXPONENTS OF K-PRIMITIVE MATRICES: THE POLYHEDRAL CONE CASE

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Abstract. Let K be a proper (i.e., closed, pointed, full convex) cone in $\mathbb{R}^n$. An $n \times n$ matrix A is said to be K-primitive if there exists a positive integer k such that $A^k(K \setminus \{0\}) \subseteq \operatorname{int} K$; the least such k is referred to as the exponent of A and is denoted by $\gamma(A)$. For a polyhedral cone K, the maximum value of $\gamma(A)$, taken over all K-primitive matrices A, is denoted by $\gamma(K)$. It is proved that for any positive integers $m, n, 3 \leq n \leq m$, the maximum value of $\gamma(K)$, as K runs through all n-dimensional polyhedral cones with m extreme rays, equals $(n-1)(m-1)+1$ when m is even or m and n are both odd, and is at least $(n-1)(m-1)$ and at most $(n-1)(m-1)+1$ when m is odd and n is even. For the cases when $m = n, m = n+1$ or $n = 3$, the cones K and the corresponding K-primitive matrices A such that $\gamma(K)$ and $\gamma(A)$ attain the maximum value are identified up to respectively linear isomorphism and cone-equivalence modulo positive scalar multiplication.

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1. Introduction

If K is a polyhedral (proper) cone in $\mathbb{R}^n$ with m extreme rays, what is the maximum value of the exponents of K-primitive matrices ? This question was posed by Steve Kirkland in an open problem session at the 8th ILAS conference held in Barcelona in July, 1999. Here by a *K-primitive matrix* we mean a real square matrix A for which there exists a positive integer k such that A^k maps every nonzero vector of K into the interior of K; the least such k is referred to as the *exponent of* A and is denoted by $\gamma(A)$. In view of Wielandt's classical sharp bound for exponents of (nonnegative) primitive matrices of a given order, Kirkland conjectured that $m^2 - 2m + 2$ is an upper bound for the maximum value considered in his question. This work is an outcome of our attempt to answer Kirkland's question.

In the classical nonnegative matrix case, the determination of upper bounds for the exponents of primitive matrices under various assumptions has been treated mainly by a graph-theoretic approach. Here for a K-primitive matrix A, we work with the digraph $(\mathcal{E}, \mathcal{P}(A, K))$, which is one of the four digraphs associated with A, introduced by Barker and Tam [9], [34]. (Formal definitions will be given later.) Based on the same digraph, Niu [24] has started an initial study of the exponents of K-primitive matrices over a polyhedral cone K. His work has motivated partly the work of Tam [33] and our present work.

The study of K-primitive matrices in the general polyhedral cone case differs from the nonnegative matrix case (or, equivalently, the simplicial cone case) in at least two (not unrelated) respects. First, in the nonnegative matrix case the (distinct) extreme vectors of the underlying cone are linearly independent, whereas in the general polyhedral cone case the extreme vectors of the underlying cone satisfy certain nonzero (linear) relations. Second, in the nonnegative matrix case it is always possible to find a nonnegative matrix with a prescribed digraph as its associated digraph, whereas in the general polyhedral cone case we often need to treat first the realization problem, that is, to determine whether there is a polyhedral cone K for which there is a K-nonnegative matrix A such that the digraph $(\mathcal{E}, \mathcal{P}(A, K))$ is given by a prescribed digraph. As expected, and also illustrated by this work, the study of the polyhedral cone case is more difficult than the classical nonnegative matrix case.

We now describe the contents of this paper in some detail.

Section 2 contains most of the definitions, together with the relevant known

results, which we need in the paper.

In Section 3 we obtain a Sedláček-Dulmage-Mendelsohn type upper bound for the local exponents (see Section 2 for the definition), and hence also an upper bound for the exponent, of a K-primitive matrix A in terms of the lengths of circuits in the digraph $(\mathcal{E}, \mathcal{P}(A, K))$ and the degree of the minimal polynomial of A.

In Section 4 we single out digraphs on $m\,(\geq 4)$ vertices, with the length of the shortest circuit equal to $m-1$, that may be realized as $(\mathcal{E}, \mathcal{P}(A, K))$ for some K-primitive matrix A, where K is a polyhedral cone with m extreme rays (see Lemma 4.1). It is found that, up to graph isomorphism, there are two of them, represented by Figure 1 and Figure 2 respectively. (These figures will be given later in the paper.) It turns out that they are precisely the two known so-called primitive digraphs on m vertices with the length of the shortest circuit equal to $m-1$. When A is a K-nonnegative matrix such that the digraph $(\mathcal{E}, \mathcal{P}(A, K))$ is given by Figure 1 or Figure 2, we make some interesting observations on A, and by delicate manipulation with the relations on the extreme vectors, we also obtain certain geometric properties of K (see Lemma 4.2). As a consequence, it is proved that if K is an n-dimensional polyhedral cone with m extreme rays then its exponent $\gamma(K)$, which is defined to be $\max\{\gamma(A) : A \text{ is } K\text{-primitive}\}$, does not exceed $(n-1)(m-1)+1$. Thus we answer in the affirmative the above-mentioned conjecture posed by Kirkland.

In Section 5 we prove that the maximum value of $\gamma(K)$ as K runs through all n-dimensional minimal cones (i.e., cones having $n+1$ extreme rays) is n^2-n+1 if n is odd, and is n^2-n if n is even. We also determine (up to linear isomorphism) the minimal cones K (and also the corresponding K-primitive matrices A) such that $\gamma(K)$ (and $\gamma(A)$) attains the maximum value. In particular, it is found that every minimal cone K whose exponent attains the maximum value has a balanced relation for its extreme vectors and also if A is a K-primitive matrix such that $\gamma(A) = \gamma(K)$ then necessarily the digraph $(\mathcal{E}, \mathcal{P}(A, K))$ is, up to graph isomorphism, given by Figure 1 or Figure 2.

Section 6 is devoted to the 3-dimensional cone case. It is proved that the maximum value of $\gamma(K)$ as K runs through all 3-dimensional polyhedral cones with m extreme rays is $2m-1$, and also that for any 3-dimensional polyhedral cone K with m extreme rays and any K-primitive matrix $A, \gamma(A) = 2m-1$ if and only if the digraph $(\mathcal{E}, \mathcal{P}(A, K))$ is, up to graph isomorphism, given by Figure 1. Indeed, for every positive integer $m \geq 3$, we can construct, for every real number $\theta \in (\frac{2\pi}{m}, \frac{2\pi}{m-1})$, a 3-dimensional polyhedral cone K_θ with m extreme rays and a K_θ-primitive matrix A_θ such that the digraph $(\mathcal{E}, \mathcal{P}(A_\theta, K_\theta))$ is given by Figure 1, and for every positive integer $m \geq 5$, we can also find a 3-dimensional polyhedral cone K with

m extreme rays for which there does not exist a K-primitive matrix A such that the digraph $(\mathcal{E}, \mathcal{P}(A,K))$ is given by Figure 1 (up to graph isomorphism). Since any two 3-dimensional polyhedral cones with the same number of extreme rays are combinatorially equivalent, this means that the exponents of combinatorially equivalent cones may not be the same. The construction of the pair (K_θ, A_θ) makes use of roots of a polynomial of the form $t^m - ct - (1-c)$, where $0 < c < 1$. In his study of Leslie matrices Kirkland ([17],[18]) has considered polynomials of a more general form, namely, those of the form $t^m - \sum_{k=1}^m a_k t^{m-k}$, where $a_1, \ldots, a_m$ are nonnegative real numbers with sum equal to 1. Evidently, there are some connections between the work of Kirkland in these two papers and our work in Section 6.

In Section 7 we show that for any positive integers $m, n, 3 \leq n \leq m$, the maximum value of $\gamma(K)$, as K runs through all n-dimensional polyhedral cones with m extreme rays, equals $(n-1)(m-1)+1$ when m is even or m and n are both odd, and is at least $(n-1)(m-1)$ and at most $(n-1)(m-1)+1$ when m is odd and n is even. Our proof involves certain generalized Vandermonde matrices, the complete symmetric polynomials, the Jacobi-Trudi determinant, and a nontrivial result about polynomials with nonnegative coefficients.

In Section 8, for the special cases when $m = n, m = n+1$ and $n = 3$, we settle the question of uniqueness of the cones K, in the class of n-dimensional polyhedral cones with m extreme rays, and the corresponding K-primitive matrices A whose exponents attain the maximum value. It is proved that for every positive integer $m \geq 5$, up to linear isomorphism, the 3-dimensional cones with m extreme rays that attain the maximum exponent are precisely the cones K_θ's introduced in Section 6, uncountably infinitely many of them; and, for $m \geq 6$, for each θ there is, up to multiples, only one K_θ-primitive matrix whose exponent attains the maximum value. In contrast, n-dimensional minimal cones whose exponents attain the maximum value are scanty — for every integer $n \geq 3$, there are (up to linear isomorphism) one or two such cones, depending on whether n is odd or even. However, for each of such minimal cones, there are uncountably infinitely many pairwise non-cone-equivalent linearly independent primitive matrices whose exponents attain the maximum value.

In Section 9, the final section, we give an example, some further remarks and a few open questions.

2. Preliminaries

We take for granted standard properties of nonnegative matrices, complex matrices and graphs that can be found in textbooks (see, for instance, [7], [8], [14], [15], [20]). A familiarity with elementary properties of finite-dimensional convex sets, convex cones and cone-preserving maps is also assumed (see, for instance,[2], [25], [31], [36]). To fix notation and terminology, we give some definitions.

Let K be a nonemtpy subset of a finite-dimensional real vector space V. The set K is called a *convex cone* if $\alpha x + \beta y \in K$ for all $x, y \in K$ and $\alpha, \beta \geq 0$; K is *pointed* if $K \cap (-K) = \{0\}$; K is *full* if its interior $\operatorname{int} K$ (in the usual topology of V) is nonempty, equivalently, $K - K = V$. If K is closed and satisfies all of the above properties, K is called a *proper cone.*

In this paper, unless specified otherwise, we always use K to denote a proper cone in the n-dimensional Euclidean space $\mathbb{R}^n$.

We denote by $\geq^K$ the partial ordering of $\mathbb{R}^n$ induced by K, i.e., $x \geq^K y$ if and only if $x - y \in K$.

A subcone F of K is called a *face* of K if $x \geq^K y \geq^K 0$ and $x \in F$ imply $y \in F$. If $S \subseteq K$, we denote by $\Phi(S)$ the *face of K generated by S*, that is, the intersection of all faces of K including S. If $x \in K$, we write $\Phi(\{x\})$ simply as $\Phi(x)$. It is known that for any vector $x \in K$ and any face F of K, $x \in \operatorname{ri} F$ if and only if $\Phi(x) = F$; also, $\Phi(x) = \{y \in K \colon x \geq^K \alpha y \text{ for some } \alpha > 0\}$. (Here we denote by $\operatorname{ri} F$ the *relative interior of F.*) A vector $x \in K$ is called an *extreme vector* if either x is the zero vector or x is nonzero and $\Phi(x) = \{\lambda x \colon \lambda \geq 0\}$; in the latter case, the face $\Phi(x)$ is called an *extreme ray.* We use $\operatorname{Ext} K$ to denote the set of all nonzero extreme vectors of K. Two nonzero extreme vectors are said to be *distinct* if they are not multiples of each other. The cone K itself and the set $\{0\}$ are always faces of K, known as *trivial faces.* Other faces of K are said to be *nontrivial*.

If S is a nonempty subset of a vector space, we denote by $\operatorname{pos} S$ the *positive hull* of S, i.e., the set of all possible nonnegative linear combinations of vectors taken from S.

A closed pointed cone K is said to be the *direct sum* of its subcones $K_1, \ldots, K_p$, and we write $K = K_1 \oplus \cdots \oplus K_p$ if every vector of K can be expressed uniquely as $x_1 + x_2 + \cdots + x_p$, where $x_i \in K_i, 1 \leq i \leq p$. K is called *decomposable* if it is the direct sum of two nonzero subcones; otherwise, it is said to be *indecomposable.* It is well-known that every closed pointed cone K can be written as

$$K = K_1 \oplus \cdots \oplus K_p,$$

where each K_j is an indecomposable cone ($1 \leq j \leq p$). Except for the ordering of the summands, the above decomposition is unique. We will refer to the K_j's as *indecomposable summands* of K.

By a *polyhedral cone* we mean a proper cone which has finitely many extreme rays. By the *dimension of a proper cone* we mean the dimension of its linear span. A polyhedral cone is said to be *simplicial* if the number of extreme rays is equal to its dimension. The nonnegative orthant $\mathbb{R}^n_+ := \{(\xi_1, \ldots, \xi_n)^T \in \mathbb{R}^n : \xi_i \geq 0 \ \forall i\}$ is a typical example of a simplicial cone.

We denote by $\pi(K)$ the set of all $n \times n$ real matrices A (identified with linear mappings on $\mathbb{R}^n$) such that $AK \subseteq K$. Members of $\pi(K)$ are said to be K-*nonnegative* and are often referred to as *cone-preserving maps*. It is clear that $\pi(\mathbb{R}^n_+)$ consists of all $n \times n$ (entrywise) nonnegative matrices.

A matrix $A \in \pi(K)$ is said to be K-*irreducible* if A leaves invariant no nontrivial face of K; A is K-*positive* if $A(K \setminus \{0\}) \subseteq \text{int } K$ and is K-*primitive* if there is a positive integer p such that A^p is K-positive. If A is K-primitive, then the smallest positive integer p for which A^p is K-positive is called the *exponent* of A and is denoted by $\gamma(A)$ (or by $\gamma_K(A)$ if the dependence on K needs to be emphasized).

It is known that the set $\pi(K)$ forms a proper cone in the space of $n \times n$ real matrices, the interior of $\pi(K)$ being the subset consisting of K-positive matrices. Also, $\pi(K)$ is polyhedral if and only if K is polyhedral. (See [31], [29] or [1].)

A matrix A is said to be *an automorphism of* K if A is invertible and A, A^{-1} both belong to $\pi(K)$ or, equivalently, $AK = K$.

It is clear that if K is a simplicial cone with n extreme rays then K is linearly isomorphic to $\mathbb{R}^n_+$. The simplicial cones may be considered as the simplest kind of cones. The next simplest kind of cones, and also the one with which we will deal considerably in this work, are the minimal cones. Minimal cones were first introduced and studied by Fiedler and Pták [12]. We call an n-dimensional polyhedral cone *minimal* if it has precisely $n+1$ extreme rays. Clearly, if K is a minimal cone with (pairwise distinct) extreme vectors $x_1, \ldots, x_{n+1}$, then (up to multiples) these vectors satisfy a unique (linear) relation. Also, a minimal cone is indecomposable if and only if the relation for its extreme vectors is full, i.e., in the relation the coefficient of each extreme vector is nonzero (see [12, Theorem 2.25]). It is readily shown that every decomposable minimal cone is the direct sum of an indecomposable minimal cone and a simplicial cone.

In dealing with (nonzero) relations on (nonzero) extreme vectors of a polyhedral

cone, we find it convenient to write such relations in the form

$$\alpha_1 x_1 + \cdots + \alpha_p x_p = \beta_1 y_1 + \cdots + \beta_q y_q,$$

where the extreme vectors $x_1, \ldots x_p, y_1, \ldots, y_q$ are pairwise distinct and the coefficients $\alpha_1, \ldots, \alpha_p, \beta_1, \ldots, \beta_q$ are all positive. Clearly we have $p, q \geq 2$.

We call a relation on extreme vectors of a polyhedral cone *balanced* if the number of nonzero terms on its two sides differ by at most 1.

An indecomposable minimal cone is said to be *of type* (p, q), where p, q are positive integers such that $2 \leq p \leq q$, if the number of (nonzero) terms on the two sides of the relation for its extreme vectors are respectively p and q. (We do not distinguish a relation with the one obtained from it by interchanging the left side with the right side.)

Given positive integers with $2 \leq p \leq q$ and $p+q = n+1$, one can construct as follows an n-dimensional indecomposable minimal cone of type (p, q). Choose any basis for $\mathbb{R}^n$, say $\{x_1, \ldots, x_n\}$, and let K be the polyhedral cone $\text{pos}\{x_1, \ldots, x_n, x_{n+1}\}$, where $x_{n+1} = (x_1 + \cdots + x_p) - (x_{p+1} + \cdots + x_n)$. Then

$$x_1 + \cdots + x_p = x_{p+1} + \cdots + x_{n+1}$$

is the (essentially) unique relation for the vectors $x_1, \ldots, x_{n+1}$. As none of the vectors $x_1, \ldots, x_{n+1}$ can be written as a nonnegative linear combination of the remaining vectors, $x_1, \ldots, x_{n+1}$ are precisely (up to nonnegative scalar multiples) all the extreme vectors of K. Therefore, K is the desired indecomposable minimal cone.

Two proper cones K_1, K_2 are said to be *linearly isomorphic* if there exists a linear isomorphism $P:\ \text{span}\, K_2 \longrightarrow \text{span}\, K_1$ such that $PK_2 = K_1$.

Using the following easy result in linear algebra, one can show that indecomposable minimal cones of the same type are linearly isomorphic.

Lemma 2.1. *Let $\{x_1, \ldots, x_k\}$ and $\{y_1, \ldots, y_k\}$ be two families of vectors in finite-dimensional vector spaces V_1 and V_2 respectively. In order that there exists a linear mapping $T : V_1 \to V_2$ that satisfies $T(x_i) = y_i$ for $i = 1, \ldots, k$, it is necessary and sufficient that $\alpha_1 y_1 + \cdots + \alpha_k y_k = 0$ is a relation for $y_1, \ldots, y_k$ whenever the corresponding relation holds for $x_1, \ldots x_k$.*

We also need the following known characterization of maximal faces of an indecomposable minimal cone ([30, Theorem 4.1]):

Theorem A. *Let K be an indecomposable minimal cone generated by extreme vectors $x_1, \ldots, x_{n+1}$ that satisfy*

$$x_1 + \cdots + x_p = x_{p+1} + \cdots + x_{n+1}.$$

Then for each pair $(i, j), 1 \le i \le p$ and $p+1 \le j \le n+1, \operatorname{pos} M_{ij}$ is a maximal face of K, where $M_{ij} = \{x_1, \ldots, x_{n+1}\} \setminus \{x_i, x_j\}$. Moreover, each maximal face of K is of this form.

Note that by the preceding theorem every maximal face, and hence every nontrivial face, of an indecomposable minimal cone is a simplicial cone in its own right.

Let $A \in \pi(K)$. In this work we need the digraph $(\mathcal{E}, \mathcal{P}(A, K))$, which is one of the four digraphs associated with A introduced by Barker and Tam ([9], [34]). It is defined in the following way: its vertex set is $\mathcal{E}$, the set of all extreme rays of K; $(\Phi(x), \Phi(y))$ is an arc whenever $\Phi(y) \subseteq \Phi(Ax)$. If there is no danger of confusion, (in particular, within proofs) we write $(\mathcal{E}, \mathcal{P}(A, K))$ simply as $(\mathcal{E}, \mathcal{P})$. It is readily checked that if K is the nonnegative orthant $\mathbb{R}^n_+$ then $(\mathcal{E}, \mathcal{P}(A, K))$ equals the usual digraph associated with A^T, the transpose of A. (If $B = (b_{ij})$ is an $n \times n$ matrix then by the usual digraph of B we mean the digraph with vertex set $\{1, \ldots, n\}$ such that (i, j) is an arc whenever $b_{ij} \neq 0$.)

It is not difficult to show that for any $A, B \in \pi(K)$, if $\Phi(A) = \Phi(B)$ then A, B are either both K-primitive or both not K-primitive, and if they are, then $\gamma(A) = \gamma(B)$. In Niu [24] it is proved that if K is a polyhedral cone then for any $A, B \in \pi(K)$, we have $(\mathcal{E}, \mathcal{P}(A, K)) = (\mathcal{E}, \mathcal{P}(B, K))$ (as labelled digraphs) if and only if $\Phi(A) = \Phi(B)$. So it is not surprising to find that the digraph $(\mathcal{E}, \mathcal{P}(A, K))$ plays a role in determining a bound for $\gamma(A)$. (When K is nonpolyhedral, the situation is more subtle. We refer the interested readers to Tam [33] for the details.)

Let K be a polyhedral cone. Then $\pi(K)$ is also polyhedral and hence has finitely many faces. Since K-primitive matrices that belong to the relative interior of the same face of $\pi(K)$ share a common exponent, it follows that there are only finitely many (integral) values that can be attained by the exponents of K-primitive matrices.

For a proper cone K, we say K *has finite exponent* if the set of exponents of K-primitive matrices is bounded; then we denote the maximum exponent by $\gamma(K)$ and refer to it as the *exponent of K*. If K has finite exponent, then a K-primitive matrix A is said to be *exp-maximal* if $\gamma(A) = \gamma(K)$. By the above discussion, every polyhedral cone has finite exponent. An example of a proper cone in $\mathbb{R}^3$ which does not have finite exponent will be given in the final section of this paper.

We will make use of the concept of a *primitive digraph*, which can be defined as a digraph for which there is a positive integer k such that for every pair of vertices i, j there is a directed walk of length k from i to j; the least such k is referred to as the *exponent of the digraph.* It is clear that a nonnegative matrix is primitive if and only if its usual digraph is primitive. It is also well-known that primitive digraphs are precisely strongly connected digraphs with the greatest common divisor of the lengths of their circuits equal to 1.

It is known that for a K-nonnegative matrix A, if A^p is K-positive then so is A^q for every $q \geq p$. This follows from the fact that if B is a K-nonnegative matrix such that $Bu \in \partial K$ for some $u \in \operatorname{int} K$ then we have $BK \subseteq \Phi(Bu) \subseteq \partial K$. The same fact also implies that the action of a K-nonnegative matrix A on a vector x in K enjoys a similar property — if $A^i x$ belongs to $\operatorname{int} K$, then so does $A^j x$ for all positive integers $j > i$.

If A is a K-nonnegative matrix and if p is a positive integer such that $A^p F \subseteq F$ for some nontrivial face F, then $A^{kp} F \subseteq F$ for all positive integers k and hence A cannot be K-primitive. This shows that the positive powers of a K-primitive matrix are all K-irreducible.

To study the exponents of K-primitive matrices, we make use of the concept of local exponent defined in the following way. (For definition of local exponent of a primitive matrix, see [8, Section 3.5].) For any K-nonnegative matrix A, not necessarily K-primitive or K-irreducible, and any $0 \neq x \in K$, by the *local exponent of* A *at* x, denoted by $\gamma(A, x)$, we mean the smallest nonnegative integer k such that $A^k x \in \operatorname{int} K$. If no such k exists, we set $\gamma(A, x)$ equal ∞. (If A is a primitive matrix and e_j is the jth standard unit vector, then $\gamma(A, e_j)$ equals the smallest integer k such that all elements in column j of A^k are nonzero.) Clearly, A is K-primitive if and only if the set of local exponents of A is bounded; in this case, $\gamma(A)$ is equal to $\max\{\gamma(A, x) : 0 \neq x \in K\}$, which is also the same as the maximum taken over all nonzero extreme vectors of K. By a compactness argument Barker [1] has shown that the K-primitivity of A is equivalent to the apparently weaker condition — which is also the definition adopted by him for K-primitivity — that all local exponents of A are finite.

By the definition of $(\mathcal{E}, \mathcal{P}(A, K))$, we have

Fact 2.2. If there is a path in $(\mathcal{E}, \mathcal{P}(A, K))$ of length k from $\Phi(x)$ to $\Phi(y)$, then $\Phi(A^k x) \supseteq \Phi(y)$.

By Fact 2.2 we obtain the following:

Fact 2.3. Let K be a polyhedral cone. If A is a K-nonnegative matrix such that the digraph $(\mathcal{E}, \mathcal{P}(A,K))$ is primitive, then A is a K-primitive matrix with exponent less than or equal to that of the primitive digraph $(\mathcal{E}, \mathcal{P}(A,K))$.

To show the preceding fact, let k denote the exponent of the primitive digraph $(\mathcal{E}, \mathcal{P})$ and let $x \in \operatorname{Ext} K$. Then there is a path of length k in the said digraph from $\Phi(x)$ to $\Phi(y)$ for any $y \in \operatorname{Ext} K$. By Fact 2.2 we have $\Phi(A^k x) \supseteq \Phi(y)$. Since this is true for every $y \in \operatorname{Ext} K$, it follows that $A^k x \in \operatorname{int} K$. But x is an arbitrary nonzero extreme vector of K, so A^k is K-positive. Hence A is K-primitive and $\gamma(A) \leq k$.

Fact 2.2 implies also the following:

Fact 2.4. Let $A \in \pi(K)$ and let $x, y \in \operatorname{Ext} K$. Suppose that $\gamma(A, y)$ is finite. If there is a path in $(\mathcal{E}, \mathcal{P}(A,K))$ of length k from $\Phi(x)$ to $\Phi(y)$, then $\gamma(A, x)$ is also finite and we have $\gamma(A, x) \leq k + \gamma(A, y)$.

Two cone-preserving maps $A_1 \in \pi(K_1)$ and $A_2 \in \pi(K_2)$ are said to be *cone-equivalent* if there exists a linear isomorphism P such that $PK_2 = K_1$ and $P^{-1}A_1P = A_2$.

Fact 2.5. Let K_1, K_2 be proper cones in $\mathbb{R}^n$. Suppose that $A_1 \in \pi(K_1)$ and $A_2 \in \pi(K_2)$ are cone-equivalent. Then:

(i) A_1 and A_2 are similar.

(ii) The cones K_1, K_2 are linearly isomorphic.

(iii) The digraphs $(\mathcal{E}, \mathcal{P}(A_1, K_1)), (\mathcal{E}, \mathcal{P}(A_2, K_2))$ are isomorphic.

(iv) Either A_1 is K_1-primitive and A_2 is K_2-primitive or they are not, and if they are, then $\gamma_{K_1}(A_1) = \gamma_{K_2}(A_2)$.

(v) For any $x \in K_2$, $\gamma(A_2, x) = \gamma(A_1, Px)$.

Also, it is clear that if K_1 and K_2 are linearly isomorphic cones, then either K_1, K_2 both have finite exponent or they both do not have, and if they both have, then $\gamma(K_1) = \gamma(K_2)$.

Under inclusion as the partial order, the set of all faces of K, denoted by $\mathcal{F}(K)$, forms a lattice with meet and join given respectively by: $F \wedge G = F \cap G$ and $F \vee G = \Phi(F \cup G)$. Two proper cones K_1, K_2 are said to be *combinatorially equivalent* if their face lattices $\mathcal{F}(K_1)$ and $\mathcal{F}(K_2)$ are isomorphic (as lattices).

For minimal cones, the concepts of “linearly isomorphic” and “combinatorially equivalent” are equivalent.

Theorem 2.6. *Let K_1, K_2 be minimal cones of dimension n_1, n_2 respectively. Suppose that for $j = 1, 2, K_j = K_j' \oplus K_j''$, where K_j' is a simplicial cone and K_j'' is an indecomposable minimal cone of type (p_j, q_j). The following conditions are equivalent:*

(i) $n_1 = n_2$ *and* $(p_1, q_1) = (p_2, q_2)$.

(ii) K_1, K_2 *are linearly isomorphic.*

(iii) K_1, K_2 *are combinatorially equivalent.*

Proof. For $j = 1, 2$, let d_j be the dimension of K_j'.

(i)⇒(ii): We have

$$d_1 = n_1 - \dim K_1'' = n_1 - (p_1 + q_1 - 1) = n_2 - (p_2 + q_2 - 1) = n_2 - \dim K_2'' = d_2;$$

so K_1' and K_2' are linearly isomorphic, being simplicial cones of the same dimension. On the other hand, K_1'' and K_2'' are also linearly isomorphic, as they are indecomposable minimal cones of the same type. Therefore, K_1 and K_2 are linearly isomorphic.

(ii)⇒(iii): Obvious.

(iii)⇒(i): As is well-known, if K is a polyhedral cone then K has faces of all possible dimensions from 0 to $\dim K$. So $\dim K$ is equal to the length of a maximal chain in the face lattice $\mathcal{F}(K)$ of K. Since the face lattices $\mathcal{F}(K_1)$ and $\mathcal{F}(K_2)$ are isomorphic, they have maximal chains of the same length. Hence, we have $n_1 = n_2$.

To proceed further, we need to establish two assertions first:

Assertion 1. Let K be an n-dimensional minimal cone, $n \geq 3$. Suppose that $K = K' \oplus K''$, where K' is a d-dimensional simplicial cone ($0 \leq d \leq n - 3$) and K'' is an $(n - d)$-dimensional indecomposable minimal cone of type (p, q). Then K has $d + pq$ maximal faces, d of which are minimal cones and the remaining are simplicial cones.

Proof. Since K is the direct sum of K' and K'', the maximal faces of K are precisely those of the form $M' \oplus K''$ or $K' \oplus M''$, where M', M'' denote respectively a maximal face of K' and a maximal face of K''. There are d maximal faces of the first kind, each of which, being the direct sum of a simplicial cone and a minimal cone, is a minimal cone in its own right. In view of Theorem A, maximal faces of the indecomposable minimal cone K'' are themselves simplicial cones and there are altogether pq of them; hence, K has pq maximal faces of the second kind, each of

which is a simplicial cone. ■

Assertion 2. A minimal cone cannot be combinatorially equivalent to a simplicial cone.

Proof. As we have already noted, any two combinatorially equivalent polyhedral cones have the same dimension. Now an n-dimensional simplicial cone has n extreme rays, whereas an n-dimensional minimal cone has $n+1$ extreme rays. So a minimal cone and a simplicial cone cannot be combinatorially equivalent. ■

Now back to the proof of the theorem. Let Ψ be a lattice isomorphism between $\mathcal{F}(K_1)$ and $\mathcal{F}(K_2)$. Clearly Ψ provides a one-to-one correspondence between the maximal faces of K_1 and those of K_2. Note that if M_1 is a maximal face of K_1 that corresponds to the maximal face M_2 of K_2, then M_1 and M_2 are combinatorially equivalent, as Ψ induces a lattice isomorphism between $\mathcal{F}(M_1)$ and $\mathcal{F}(M_2)$. In view of Assertion 2, under Ψ, maximal faces of K_1 which are themselves minimal (respectively, simplicial) cones correspond to maximal faces of K_2 which are themselves minimal (respectively, simplicial) cones. By Assertion 1, we have $d_1 = d_2$ and $p_1q_1 = p_2q_2$. But we have already shown that $n_1 = n_2$ and also we have $p_j + q_j = n_j - d_j + 1$ for $j = 1, 2$, it follows that we have $(p_1, q_1) = (p_2, q_2)$. ■

3. Upper bounds for exponents

Hereafter, for every pair of positive integers m, n with $3 \leq n \leq m$, we denote by $\mathcal{P}(m, n)$ the set of all n-dimensional polyhedral cones with m extreme rays. Note that we start with $n = 3$ as the cases $n = 1$ or 2 are trivial. Also, when $m = 3$, in order that $\mathcal{P}(m, n)$ is nonvacuous, n must be 3.

The following theorem of Sedláček [27] and Dulmage and Mendelsohn [10] (see, for instance, [8, Theorem 3.5.4]) gives an upper bound for the exponent of a primitive matrix A in terms of lengths of circuits in the digraph of A.

Theorem B. *Let A be an $n \times n$ primitive matrix. If s is the length of the shortest circuit in the digraph of A, then $\gamma(A) \leq n + s(n-2)$.*

By setting $s = n - 1$ in Theorem B, one recovers the sharp general upper bound $(n-1)^2 + 1$, due to Wielandt [35], for exponents of $n \times n$ primitive matrices.

The next lemma gives an analogous result on the local exponents of a cone-preserving map, which is essential to our work.

If D is a digraph, v is a vertex of D and W is a nonempty subset of the vertex set of D, then we say v *has access to* W if there is a path from v to a vertex of W. In this case, the length (i.e., the number of edges) of the shortest path from v to a vertex of W is referred to as the *distance* from v to W. If v belongs to W, the distance is taken to be zero.

For a square matrix C, we denote by m_C the *degree of the minimal polynomial of* C.

Lemma 3.1. *Let K be a proper cone and let $A \in \pi(K)$. Let $\Phi(x)$ be a vertex of $(\mathcal{E}, \mathcal{P}(A,K))$ which is at a distance $w\,(\geq 0)$ to a circuit $\mathcal{C}$ of length l. Suppose that A^l is K-irreducible, or that the circuit $\mathcal{C}$ contains a vertex $\Phi(u)$ for which $\gamma(A,u)$ is finite. Then $\gamma(A,x)$ is finite and*

$$\gamma(A,x) \leq w + (m_{A^l} - 1)l \leq w + (m_A - 1)l \leq w + (n-1)l.$$

Proof. Let $\mathcal{C}\colon \Phi(x_1) \to \Phi(x_2) \to \cdots \to \Phi(x_l) \to \Phi(x_1)$ be the circuit under consideration. (Here, for convenience, we represent the arc $(\Phi(x), \Phi(y))$ by $\Phi(x) \to \Phi(y)$.) Without loss of generality, we may assume that the distance from $\Phi(x)$ to $\Phi(x_1)$ is w. Since there is a path of length l from $\Phi(x_1)$ to itself, by Fact 2.2 we have $\Phi(A^l x_1) \supseteq \Phi(x_1)$, which implies the following chain of inclusions:

$$\Phi(x_1) \subseteq \Phi(A^l x_1) \subseteq \Phi(A^{2l} x_1) \subseteq \cdots \subseteq \Phi(A^{jl} x_1) \subseteq \Phi(A^{(j+1)l} x_1) \subseteq \cdots.$$

Let p denote the dimension of the subspace $\operatorname{span}\{(A^l)^j x_1 : j = 0, 1, \ldots\}$. By the above chain of inclusions, clearly the face $\Phi((A^l)^{p-1} x_1)$ contains the vectors $x_1, A^l x_1, \ldots, (A^l)^{p-1} x_1$, which are linearly independent and hence form a basis for the said subspace; so $\Phi((A^l)^{p-1} x_1)$ includes, and hence is equal to, $\Phi(\operatorname{span}\{(A^l)^j x_1 : j = 0, 1, \ldots\} \cap K)$. Note that the latter face is the smallest A^l-invariant face of K that contains x_1. If A^l is K-irreducible, the latter face is clearly K. On the other hand, if the circuit $\mathcal{C}$ contains a vertex $\Phi(u)$ for which $\gamma(A,u)$ is finite, then by Fact 2.4 $\gamma(A, x_1)$ is also finite. Hence $A^j x_1 \in \operatorname{int} K$ for all positive integers j sufficiently large and, as a consequence, the smallest A^l-invariant face of K that contains x_1 is K. In either case, we have, $\Phi((A^l)^{p-1} x_1) = K$ and so $(A^l)^{p-1} x_1 \in \operatorname{int} K$; hence $\gamma(A, x_1) \leq (p-1)l$. Then by Fact 2.4 we have

$$\gamma(A,x) \leq w + \gamma(A,x_1) \leq w + (p-1)l.$$

It is clear that $p \le m_{A^l}$. But we also have $m_{A^l} \le m_A \le n$, so the desired inequalities follow. ■

Lemma 3.2. *Let $K \in \mathcal{P}(m,n)$ and let A be a K-primitive matrix. If $\Phi(x)$ is a vertex of $(\mathcal{E}, \mathcal{P}(A,K))$ which has access to a circuit of length l, then*

$$\gamma(A,x) \le m + (m_A - 2)l \le m + (n-2)l.$$

Proof. This follows from Lemma 3.1, as the distance from $\Phi(x)$ to $\mathcal{C}$ is at most $m - l$ and also A^l is K-irreducible. ■

Using Lemma 3.1, one can also readily deduce the following result.

Corollary 3.3. *Let $K \in \mathcal{P}(m,n)$, and let $A \in \pi(K)$. Suppose that the digraph $(\mathcal{E}, \mathcal{P}(A,K))$ is strongly connected. Let s be the shortest circuit length of the digraph. If A^s is K-irreducible, then A is K-primitive and $\gamma(A) \le m + s(m_A - 2)$.*

It is known that if K is a polyhedral cone with m extreme rays, then a K-nonnegative matrix A is K-primitive if A^j are K-irreducible for $j = 1, \ldots, 2^m - 1$ (see [1, Theorem 2]). The preceding corollary tells us that when the digraph $(\mathcal{E}, \mathcal{P})$ is strongly connected, to show the K-primitivity of A, it suffices to check the K-irreducibility of only one positive power of A.

Clearly, the following result of Niu [Niu] is a consequence of Corollary 3.3:

Theorem C. *Let $K \in \mathcal{P}(m,n)$ and let A be K-primitive. If the digraph $(\mathcal{E}, \mathcal{P}(A,K))$ is strongly connected and s is the length of the shortest circuit in $(\mathcal{E}, \mathcal{P}(A,K))$, then $\gamma(A) \le m + s(m-2)$.*

In Theorem C, by choosing $K = \mathbb{R}^n_+$ we recover Theorem B.

It is known (see [29]) that if A is K-irreducible, then $(I+A)^{n-1}$ is K-positive (where n is the dimension of K). Hartwig and Neumann [16] have shown that in the nonnegative matrix case the result can be strengthened by replacing n by m_A, the degree of the minimal polynomial of A. Now we can show that the latter improvement is also valid for a cone-preserving map on a proper cone.

Corollary 3.4. *If $A \in \pi(K)$ is K-irreducible, then $(I+A)^{m_A - 1}$ is K-positive.*

Proof. If A is K-irreducible, then clearly $I + A$ is also K-irreducible and in the digraph $(\mathcal{E}, \mathcal{P}(I+A,K))$ there is a loop at each vertex. By Lemma 3.1, $\gamma(I+A,x) \le m_{I+A} - 1$ for every $x \in \operatorname{Ext} K$. But $m_{I+A} = m_A$, so $(I+A)^{m_A-1}$ is

K-positive. ■

It is also possible to provide a direct proof for Corollary 3.4, one that does not involve the digraph $(\mathcal{E}, \mathcal{P})$.

We denote by $\mathcal{N}(A)$ the nullspace of A. It is easy to show that for any $A \in \pi(K), \mathcal{N}(A) \cap K = \{0\}$ if and only if the outdegree of each vertex of $(\mathcal{E}, \mathcal{P})$ is positive. As a consequence, for any K-primitive matrix A, the digraph $(\mathcal{E}, \mathcal{P})$ has at least one circuit.

In contrast with the nonnegative matrix case, the digraph $(\mathcal{E}, \mathcal{P})$ associated with a K-primitive matrix A (where K is polyhedral) need not be strongly connected. (Many such examples can be found in [33].) Nevertheless, every vertex of $(\mathcal{E}, \mathcal{P})$ has access to some circuit of $(\mathcal{E}, \mathcal{P})$. This makes it possible to apply Lemma 3.2 to obtain bounds for the exponents of K-primitive matrices.

As yet another application of Lemma 3.1, we obtain the following result, which is an extension of the corresponding result for a symmetric primitive matrix (cf. [8, Theorem 3.5.3]). Recall that a digraph D is said to be *symmetric* if for every pair of vertices u, v of D, (u, v) is an arc if and only if (v, u) is an arc.

Corollary 3.5. *Let $A \in \pi(K)$. If A is K-primitive and the digraph $(\mathcal{E}, \mathcal{P}(A, K))$ is symmetric, then*

$$\gamma(A) \leq 2(m_{A^2} - 1) \leq 2(m_A - 1).$$

Proof. Since A is K-primitive, $\mathcal{N}(A) \cap K = \{0\}$; hence the digraph $(\mathcal{E}, \mathcal{P})$ has an outgoing edge (possibly a loop) at each vertex. As the digraph $(\mathcal{E}, \mathcal{P})$ is symmetric, it follows that $(\mathcal{E}, \mathcal{P}(A^2, K))$ has a loop at each vertex. By Lemma 3.1 we have $\gamma(A^2) \leq m_{A^2} - 1$ and hence $\gamma(A) \leq 2(m_{A^2} - 1) \leq 2(m_A - 1)$. ■

It is clear that for any K-primitive matrix $A, m_A \geq 2$. When $m_A = 2$, more can be said.

Lemma 3.6. *Let A be K-primitive. If $m_A = 2$ then $\gamma(A) = 1$ or 2.*

Proof. Since $m_A = 2$ (and A is a real matrix), there exist real numbers a, b such that $A^2 + aA + bI = 0$. Clearly, a, b cannot be both zero, as A is not nilpotent. By the pointedness of the cone $\pi(K)$, at least one of a, b is negative. If $b < 0$ and $a \geq 0$, then A^2 belongs to the face $\Phi(I)$ (of $\pi(K)$) and so it must be K-reducible, which is a contradiction. If $a < 0$ and $b \geq 0$, then $A^2 \in \Phi(A)$ or $A^2 \leq \alpha A$ for some $\alpha > 0$, which implies that all positive powers of A lie in $\Phi(A)$. But A^p is K-positive

or, equivalently, belongs to $\operatorname{int}\pi(K)$ for p sufficiently large, it follows that in this case we must have $\Phi(A) = \pi(K)$, or in other words, $\gamma(A) = 1$. In the remaining case when a, b are both negative, A^2 is a positive linear combination of A and I and hence lies in $\operatorname{ri}\Phi(A+I)$. Then one readily shows that all positive powers of A also lie in $\operatorname{ri}\Phi(A+I)$. By the K-primitivity of A, A^p belongs to $\operatorname{int}\pi(K)$ for p sufficiently large. This implies that $\Phi(A+I) = K$. As A^2 is a positive linear combination of A and I, it follows that A^2 also belongs to $\operatorname{int}\pi(K)$; so we have $\gamma(A) \leq 2$. This completes the proof. ∎

4. Special digraphs for K-primitive matrices

The results of Section 3 may suggest that for a K-primitive matrix A the longer the shortest circuit in $(\mathcal{E}, \mathcal{P})$ is, the larger is the value of $\gamma(A)$. Given a pair of positive integers m, n with $3 \leq n \leq m$, what can we say about digraphs on m vertices, with the length of the shortest circuit equal to $m-1$, which can be identified (up to graph isomorphism) with $(\mathcal{E}, \mathcal{P}(A, K))$ for some pair (A, K) where $K \in \mathcal{P}(m, n)$ and A is a K-primitive matrix ? It turns out that such digraphs must be primitive. When $m \geq 4$, apart from the labelling of its vertices (or, in other words, up to graph isomorphism), there are two such digraphs, which are given by Figure 1 and Figure 2. When $m = 3$, there is one more digraph. (See Lemma 4.1 below.)

Note that if K is a polyhedral cone with m extreme rays then for any K-primitive matrix A, the length of the shortest circuit in $(\mathcal{E}, \mathcal{P})$ is at most $m-1$. This is because, if the length of the shortest circuit is m, then the digraph must be a circuit of length m and, as a consequence, A^m is K-reducible, which is impossible.

In what follows when we say the digraph $(\mathcal{E}, \mathcal{P})$ is given by Figure 1 (or by other figures), we mean the digraph is given either by the figure up to graph isomorphism or by the figure as a labelled digraph. In most instances, we mean it in the former sense but in a few instances we mean it in the latter sense. It should be clear from the context in what sense we mean. (For instance, in parts (i) and (iii) of Lemma 4.2 we mean the former sense, but in part (ii) we mean the latter sense.)

We will obtain certain geometric properties of K when K is a non-simplicial polyhedral cone with $m\,(\geq 4)$ extreme rays for which there exists a K-primitive matrix A such that $(\mathcal{E}, \mathcal{P})$ is given by Figure 1 or Figure 2. More precisely, we show that if $(\mathcal{E}, \mathcal{P})$ is given by Figure 1 then K is indecomposable; if $(\mathcal{E}, \mathcal{P})$ is given by Figure 2 and if K is decomposable, then m is odd and K is the direct sum of a

ray and an indecomposable minimal cone with a balanced relation on its extreme vectors. We also obtain some properties on the corresponding K-primitive matrix A. It will be shown that if K is a polyhedral cone with m extreme rays, then for any K-primitive matrix A, $\gamma(A) \leq (m_A - 1)(m-1) + 1$, where the equality holds only if the digraph $(\mathcal{E}, \mathcal{P})$ is given by Figure 1. As a consequence, $(n-1)(m-1)+1$ is an upper bound for $\gamma(K)$ when $K \in \mathcal{P}(m, n)$.

Lemma 4.1. *Let $K \in \mathcal{P}(m, n)$ $(3 \leq n \leq m)$ and let A be a K-primitive matrix. Then the length of the shortest circuit in the digraph $(\mathcal{E}, \mathcal{P}(A, K))$ equals $m-1$ if and only if the digraph $(\mathcal{E}, \mathcal{P}(A, K))$ is, apart from the labelling of its vertices, given by Figure 1 or Figure 2, or (in case $m = n = 3$) by the digraph of order 3 whose arcs are precisely all possible arcs between every pair of distinct vertices:*

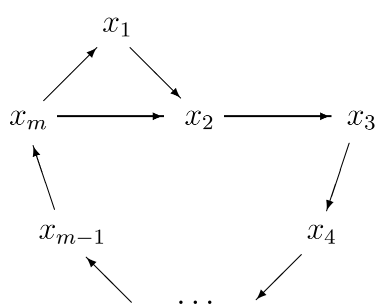

Figure 1.

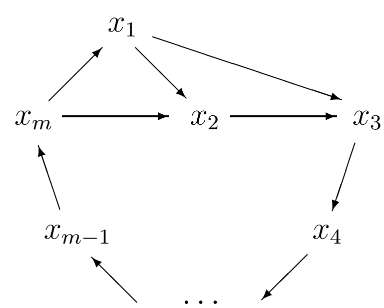

Figure 2.

(For simplicity, we label the vertex $\Phi(x_i)$ simply by x_i.).

Proof. We treat only the "only if" part, the "if" part being obvious.

It is not difficult to show that there are precisely three non-isomorphic primitive digraphs of order three with shortest circuit length two, namely, the digraphs given by Figure 1, Figure 2 and the one with all possible arcs between every pair of distinct vertices. So there is no problem when $m = 3\,(= n)$. Hereafter, we assume that $m \geq 4$.

Let $x_1, \dots, x_m$ denote the pairwise distinct extreme vectors of K. Let A be a K-primitive matrix such that the length of the shortest circuit in the digraph $(\mathcal{E}, \mathcal{P})$ is $m-1$. Without loss of generality, we may assume that the digraph $(\mathcal{E}, \mathcal{P})$ contains the circuit $\mathcal{C}$ (of length $m-1$) that is made up of the arcs $(\Phi(x_m), \Phi(x_2))$ and $(\Phi(x_j), \Phi(x_{j+1}))$ for $j = 2, 3, \dots, m-1$. Being a circuit of shortest length, $\mathcal{C}$ cannot contain any chord, nor can it have loops at its vertices. If there is no arc from a vertex of $\mathcal{C}$ to the remaining vertex $\Phi(x_1)$, then we have $A\Phi(x_m) = \Phi(x_2)$ and

$A\Phi(x_j) = \Phi(x_{j+1})$ for $j = 2, 3, \ldots, m-1$ and it will follow that $A^{m-1}x_m$ is a positive multiple of x_m, hence A^{m-1} is K-reducible, which contradicts the assumption that A is K-primitive. So there is at least one arc from a vertex of $\mathcal{C}$ to $\Phi(x_1)$, say, $(\Phi(x_m), \Phi(x_1))$ is one such arc. Similarly, there is also an arc from $\Phi(x_1)$ to a vertex of $\mathcal{C}$. Since the length of the shortest circuit in the digraph $(\mathcal{E}, \mathcal{P})$ is $m-1$, there cannot be an arc of the form $(\Phi(x_1), \Phi(x_j))$ with $4 \le j \le m$. So $(\Phi(x_1), \Phi(x_2))$ and $(\Phi(x_1), \Phi(x_3))$ are the only possible arcs with initial vertex $\Phi(x_1)$, and at least one of them must be present.

We treat the case when the arc $(\Phi(x_1), \Phi(x_2))$ is present first. Clearly, none of the arcs $(\Phi(x_j), \Phi(x_1))$, for $j = 2, \ldots, m-2$, can be present, as the length of the shortest circuit in $(\mathcal{E}, \mathcal{P})$ is $m-1$. However, it is possible that $(\Phi(x_{m-1}), \Phi(x_1))$ is an arc, provided that $(\Phi(x_1), \Phi(x_3))$ is not an arc. Note that the digraph that consists of the circuit $\mathcal{C}$ and the arcs $(\Phi(x_m), \Phi(x_1)), (\Phi(x_1), \Phi(x_2)), (\Phi(x_{m-1}), \Phi(x_1))$ is isomorphic with the one given by Figure 2. So, in this case, up to graph isomorphism, the digraph $(\mathcal{E}, \mathcal{P})$ is given by Figure 1 or Figure 2.

Now we consider the case when the arc $(\Phi(x_1), \Phi(x_2))$ is absent. Then the arc $(\Phi(x_1), \Phi(x_3))$ is present and the digraph $(\mathcal{E}, \mathcal{P})$ contains two circuits of length $m-1$. As each of these two circuits cannot contain a chord, $(\Phi(x_2), \Phi(x_1))$ is the only possible remaining arc. If the arc $(\Phi(x_2), \Phi(x_1))$ is absent, then a little calculation shows that $A^{m-1}x_m$ is a positive multiple of x_m, which violates the assumption that A is K-primitive. On the other hand, if the arc $(\Phi(x_2), \Phi(x_1))$ is present, then the digraph $(\mathcal{E}, \mathcal{P})$ is isomorphic with the one given by Figure 2. This completes the proof. ■

Note that Figure 1 is the same as the (unique) digraph associated with an $m \times m$ primitive matrix whose exponent attains Wielandt's bound $m^2 - 2m + 2$ (see [8]).

To proceed further, we need to manipulate with the relations on the extreme vectors of a polyhedral cone. We now explain the relevant terminology.

Let R be a relation on the extreme vectors of a polyhedral cone K. Suppose that the vectors that appear in R come from $p\,(\ge 2)$ different indecomposable summands of K, say, $K_1, \ldots, K_p$. To be specific, let R be given by: $\sum_{i \in M} \alpha_i x_i = \sum_{j \in N} \beta_j y_j$, where M, N are finite index sets, each with at least two elements and the α_is, β_js are all positive real numbers. For each $r = 1, \ldots, p$, let $M_r = \{i \in M : x_i \in K_r\}$ and $N_r = \{j \in N : y_j \in K_r\}$. Then for each fixed r, rewriting relation R, we obtain

$$\sum_{i \in M_r} \alpha_i x_i - \sum_{j \in N_r} \beta_j y_j = \sum_{j \in N \setminus N_r} \beta_j y_j - \sum_{i \in M \setminus M_r} \alpha_i x_i.$$

Now the vector on the left side of the above relation belongs to span K_r, whereas the

one on the right side belongs to $\sum_{s\neq r} \operatorname{span} K_s$. But $\operatorname{span} K_r \cap \sum_{s\neq r} \operatorname{span} K_s = \{0\}$ (as $K_1, \ldots, K_p$ are pairwise distinct indecomposable summands of K), so it follows that we have the relation

$$\sum_{i\in M_r} \alpha_i x_i = \sum_{j\in N_r} \beta_j y_j,$$

which we denote by R_r. This is true for each r. It is clear that relation R can be obtained by adding up relations $R_1, \ldots, R_p$. In this case, we say relation R *splits into the subrelations* $R_1, \ldots, R_p$. Note that each R_r has at least four (nonzero) terms. So when we pass from the relation R to one of its subrelations R_r, the number of terms involved in the relation decreases by at least four.

Recall that an $n\times n$ complex matrix A is said to be *non-derogatory* if every eigenvalue of A has geometric multiplicity 1 or, equivalently, if the minimal and characteristic polynomials of A are identical. (See, for instance, [15, Theorem 3.3.15].)

Lemma 4.2. *Let $K \in \mathcal{P}(m,n)$ $(3 \le n \le m)$. Let A be a K-nonnegative matrix. Suppose that the digraph $(\mathcal{E}, \mathcal{P}(A,K))$ is given by Figure 1 or Figure 2. Then:*

(i) *A is K-primitive, nonsingular, non-derogatory, and has a unique annihilating polynomial of the form $t^m - ct - d$, where $c, d > 0$.*

(ii) *$\gamma(A)$ equals $\gamma(A, x_1)$ or $\gamma(A, x_2)$ depending on whether the digraph $(\mathcal{E}, \mathcal{P}(A,K))$ is given by Figure 1 or Figure 2. In either case, $\max_{1\le i\le m} \gamma(A, x_i)$ is attained at precisely one i.*

(iii) *Assume, in addition, that K is non-simplicial. If $(\mathcal{E}, \mathcal{P}(A,K))$ is given by Figure 1 then K must be indecomposable. If the digraph is given by Figure 2 then either K is indecomposable or m is odd and K is the direct sum of a ray and an indecomposable minimal cone with a balanced relation for its extreme vectors.*

Proof. (i) Since the digraphs given by Figure 1 and Figure 2 are primitive, by Fact 2.3 A is K-primitive. To show that A is nonsingular, we treat the case when the digraph $(\mathcal{E}, \mathcal{P})$ is given by Figure 1, the argument for the other case being similar. Then for $j = 1, \ldots, m-1$, Ax_j is a positive multiple of x_{j+1}. So $x_2, x_3, \ldots, x_m$ all belong to $\mathcal{R}(A)$, the range space of A. On the other hand, since $x_2 \in \mathcal{R}(A)$ and Ax_m is a positive linear combination of x_1 and x_2, we also have $x_1 \in \mathcal{R}(A)$. Therefore, regarded as a linear map A is onto and hence is nonsingular.

To establish the second half of this part, we may assume that $\rho(A) = 1$ as $\rho(A) > 0$, A being K-primitive. We first deal with the case when $(\mathcal{E}, \mathcal{P}(A,K))$ is

given by Figure 1. Since A is K-primitive, A^T is K^*-primitive. Let v denote the Perron vector of A^T. As $v \in \operatorname{int} K^*$, $C := \{x \in K : \langle x, v \rangle = 1\}$ is a complete (and hence compact) cross-section of K and indeed it is a polytope with m extreme points. Replacing the extreme vectors $x_1, \ldots, x_m$ of K by suitable positive multiples, we may assume that $x_1, \ldots, x_m$ are precisely all the extreme points of C. It is clear that $AC \subseteq C$, as $A \in \pi(K)$ and $\rho(A) = 1$. Since the digraph $(\mathcal{E}, \mathcal{P})$ is given by Figure 1, we have $Ax_j = x_{j+1}$ for $j = 1, \ldots, m-1$ and also $Ax_m = (1-c)x_1 + cx_2$ for some $c \in (0,1)$. The latter condition can be rewritten as $(A^m - cA - (1-c)I_n)x_1 = 0$. It is clear that the A-invariant subspace of $\mathbb{R}^n$ generated by x_1 is $\mathbb{R}^n$ itself; so A is non-derogatory and also it follows that $t^m - ct - (1-c)$ is an annihilating polynomial for A. Therefore, A has an annihilating polynomial of the desired form.

When $(\mathcal{E}, \mathcal{P}(A, K))$ is given by Figure 2, we proceed in a similar way. In this case we have

$$Ax_i = x_{i+1} \text{ for } i = 2, \ldots, m-1,$$

and

$$Ax_m = (1-a)x_1 + ax_2 \text{ and } Ax_1 = (1-b)x_2 + bx_3$$

for some $a, b \in (0,1)$. Then after a little calculation we obtain

$$[A^m - ((1-a)b + a)\, A - (1-a)(1-b)I]x_2 = 0.$$

Since the A-invariant subspace of $\mathbb{R}^n$ generated by x_2 is $\mathbb{R}^n$ itself, it follows that A is non-derogatory and $t^m - ((1-a)b + a)t - (1-a)(1-b)$ is an annihilating polynomial for A. The latter polynomial can be rewritten as $t^m - ct - (1-c)$, where $c = (1-a)b + a \in (0,1)$.

The uniqueness of the annihilating polynomial for A of the desired form is obvious, because $\{A, I_n\}$ is a linearly independent set.

(ii) Note that for any $0 \neq x \in K$ and $j = 0, 1, \ldots, \gamma(A, x) - 1, \gamma(A, x) = \gamma(A, A^j x) + j$.

First, consider the case when the digraph $(\mathcal{E}, \mathcal{P})$ is given by Figure 1. Since $A^{j-1}x_1$ is a positive multiple of x_j for $j = 2, \ldots, m$, we have $\gamma(A, x_1) > m$ and

$$\gamma(A, x_j) = \gamma(A, A^{j-1}x_1) = \gamma(A, x_1) - j + 1$$

for $j = 2, \ldots, m$; hence

$$\gamma(A) = \max_{1 \leq j \leq m} \gamma(A, x_j) = \gamma(A, x_1).$$

When the digraph $(\mathcal{E}, \mathcal{P})$ is given by Figure 2, we apply a similar but slightly more elaborate argument. For $j = 3, \ldots, m$, we have

$$\gamma(A, x_2) = \gamma(A, A^{j-2}x_2) + j - 2 = \gamma(A, x_j) + j - 2;$$

hence $\gamma(A, x_2) > \gamma(A, x_j)$ for each such j. A little calculation shows that $A^m x_2$ is a positive linear combination of x_2 and x_3. But Ax_1 is also a positive linear combination of x_2 and x_3, hence $\Phi(Ax_1) = \Phi(A^m x_2)$. So we have

$$\gamma(A, x_2) = \gamma(A, A^m x_2) + m = \gamma(A, Ax_1) + m = \gamma(A, x_1) - 1 + m,$$

which implies $\gamma(A, x_2) > \gamma(A, x_1)$. Therefore, we have $\gamma(A) = \gamma(A, x_2)$.

(iii) In the following argument, unless specified otherwise, we assume that the digraph $(\mathcal{E}, \mathcal{P})$ is given by Figure 1 or Figure 2. First, we show that each of the extreme vectors $x_1, \ldots, x_m$, except possibly x_2, is involved in at least one relation on $\operatorname{Ext} K$. For the purpose, it suffices to show that x_3 is involved in one such relation; because, by applying A or its positive powers to a relation on $\operatorname{Ext} K$ involving x_3, we can obtain for each of the vectors $x_4, \ldots, x_{m-1}, x_m, x_1$ a relation that involves the vector. Suppose that x_3 is not involved in any (nonzero) relation on $\operatorname{Ext} K$. Take any relation S on $\operatorname{Ext} K$; as K is non-simplicial, such relation certainly exists. Note that, since x_3 does not appear in S, x_4 (and also x_3) cannot appear in the (necessarily nonzero) relation obtained from S by applying A. Similarly, the vectors x_3, x_4, x_5 all do not appear in the relation obtained from S by applying A^2. Continuing the argument, we can show that the only vectors that can appear in the nonzero relation obtained from S by applying A^{m-3} are x_1, x_2. This contradicts the hypothesis that x_1, x_2 are distinct nonzero extreme vectors of K.

Next, we note that if x_2 is not involved in any relation on $\operatorname{Ext} K$ and if $(\mathcal{E}, \mathcal{P})$ is given by Figure 1 then, by applying A repeatedly to a nonzero relation on $\operatorname{Ext} K$ sufficiently many times, we would obtain a nonzero relation on $\operatorname{Ext} K$ that involves less than four vectors, which is a contradiction. So if x_2 is not involved in any relation on $\operatorname{Ext} K$ then $(\mathcal{E}, \mathcal{P})$ must be given by Figure 2.

Now we contend that K is either indecomposable or is the direct sum of a ray and an indecomposable cone. By what we have done above, each of the extreme vectors $x_1, \ldots, x_m$, except possibly x_2, belongs to an indecomposable summand of K that is not a ray. Let K_1 be the indecomposable summand of K that contains x_m. To establish our contention, it remains to show that K has no indecomposable summand which is not a ray and is different from K_1. It is known that every indecomposable polyhedral cone of dimension greater than 1 has a full relation for

its extreme vectors (see [12], p.37, (2.14)). So it suffices to show that there is no relation on $\operatorname{Ext} K$ that involves vectors not belonging to K_1. Assume to the contrary that there are such relations. Let T_0 be one such shortest relation (i.e., one having the minimum number of terms). Note that, since x_m is not involved in T_0, the relation obtained from T_0 by applying A has the same number of terms as T_0, unless x_1 appears in T_0 but x_2 does not and $(\mathcal{E}, \mathcal{P})$ is given by Figure 2, in which case the said relation may have one term more than T_0. Suppose that the extreme vectors that appear in T_0 are $x_{k_1}, x_{k_2}, \ldots, x_{k_s}$, where $1 \leq k_1 < k_2 < \cdots < k_s \leq m-1$. It is readily seen that the relations obtained from T_0 by applying A^i for $i = 1, \ldots, m-k_s$ all have the same number of terms, as x_1 and x_m are both not involved in the first $m - k_s - 1$ of these relations.

Let q denote the least positive integer such that $x_{m-q} \notin K_1$. Certainly, we have $k_s \leq m-q$. Denote by $\tilde{T_0}$ the relation obtained from T_0 by applying A^{m-q-k_s}. (If $k_s = m-q$, we take $\tilde{T_0}$ to be T_0.) Then $\tilde{T_0}$ either has the same number of terms as T_0 or has one term more. Note that now x_{m-q} is involved in $\tilde{T_0}$ and, by our choice of q, $x_{m-q} \notin K_1$. If $\tilde{T_0}$ involves also extreme vectors of K_1, then $\tilde{T_0}$ splits and we would obtain a relation for extreme vectors not belonging to K_1, which has at least four terms fewer than that of $\tilde{T_0}$ and hence is a relation shorter than the shortest relation T_0, which is a contradiction. So we assume that all the vectors appearing in $\tilde{T_0}$ do not belong to K_1 (and in fact they all lie in the same indecomposable summand of K).

For $j = 1, 2, \ldots$, let T_j denote the relation obtained from $\tilde{T_0}$ by applying A^j. By considering the cases when $\tilde{T_0} = T_0$ and $\tilde{T_0} \neq T_0$ separately, one readily sees that relation T_1 has at most one term more than T_0. Also, T_1 involves x_{m-q+1} which, by the definition of q, belongs to K_1. If T_1 involves also extreme vectors not belonging to K_1, then T_1 splits and we would obtain a contradiction. So T_1 is a relation on $\operatorname{Ext} K_1$ and $x_{\tilde{k}_1+1}, \ldots, x_{\tilde{k}_s+1}$ all belong to K_1, where $\tilde{k}_j = (m-q-k_s) + k_j$. By the same argument we may assume that for $j = 1, \ldots, q, T_j$ is a relation on $\operatorname{Ext} K_1$ with at most one term more than T_0. So, we have $x_{\tilde{k}_j+r} \in K_1$ for $r = 1, \ldots, q$ and $j = 1, \ldots, s$. Note that x_m is involved in T_q but x_1 is not (as x_m is not involved in T_{q-1}). So x_1, x_2 are both involved in T_{q+1} and lie on the same side of it. As a consequence, x_2, x_3 are both involved in T_{q+2}, x_3, x_4 are both involved in T_{q+3}, and so forth. Clearly, T_{q+1} has one term more than T_q and hence at most two terms more than T_0.

If T_{q+1} involves vectors not belonging to the same indecomposable summand of K, then the relation splits and the minimality of T_0 would be violated. So we assume

that T_{q+1} is a relation on $\operatorname{Ext} K_2$ — here K_2 may or may not be the same as K_1. Note that now we have $x_1, x_2, x_{\tilde{k}_j+q+1} \in K_2$ for $j = 1, \ldots, s-1$. Let l denote the smallest positive integer such that at least one of the vectors $A^l x_2, A^l x_{\tilde{k}_j+q+1}, j = 1, \ldots, s-1$ does not belong to K_2. It is readily seen that $A^l x_2$ is always a positive multiple of x_{2+l} and $A^l x_1$ is a positive multiple of x_{1+l} or a positive linear combination of x_{1+l} and x_{2+l}, depending on whether $(\mathcal{E}, \mathcal{P})$ is given by Figure 1 or Figure 2. If $A^l x_2 \notin K_2$, then, from the definition of l, necessarily we have $x_{2+l} \notin K_2$ and $x_{1+l} \in K_2$. On the other hand, if $A^l x_2 \in K_2$, then there must exist $j = 1, \ldots, s-1$ such that $A^l x_{\tilde{k}_j+q+1} \notin K_2$. In any case, the relation obtained from T_{q+1} by applying A^l involves at least one extreme vector in K_2 and at least one extreme vector not in K_2. Then the relation splits and we would obtain a relation for extreme vectors not belonging to K_1, which is shorter than the shortest such relation, which is a contradiction. (A cautious reader may wonder whether there is a positive integer $r < l$ such that $A^r x_{\tilde{k}_{s-1}+q+1} = x_m$. If this is so, then at one of the steps in applying A l times to relation T_{q+1} the number of terms in the relation is increased by one. However, it is possible to show that such positive integer r does not exist by analyzing carefully the cases $\tilde{k}_{s-1} + q + 1 < \tilde{k}_s$ and $\tilde{k}_{s-1} + q + 1 = \tilde{k}_s$ separately.)

It remains to show that if $(\mathcal{E}, \mathcal{P})$ is given by Figure 2 and K is the direct sum of the ray $\operatorname{pos}\{x_2\}$ and the indecomposable polyhedral cone $\operatorname{pos}\{x_1, x_3, x_4, \ldots, x_m\}$, then m is odd and the latter cone is an indecomposable minimal cone with a balanced relation for its extreme vectors. Note that the assumption that x_2 does not appear in any relation on $\operatorname{Ext} K$ guarantees that every relation obtained from a shortest relation on $\operatorname{Ext} K$ by applying A or its positive powers is still a shortest relation. We contend every shortest relation involves each of the vectors $x_1, x_3, \ldots, x_m$. Assume that the contrary holds. Take a shortest relation R. Since R has at least four terms, one of the vectors $x_3, x_4, \ldots, x_m$ must appear in R. On the other hand, R cannot involve all of these vectors; otherwise, x_1 does not appear in R, and so the relation obtained from R by applying A involves the vector x_2, which is a contradiction. Thus we can find an $i, 4 \leq i \leq m$, such that x_i appears in R and x_{i-1} does not or the other way round. Then the relation obtained from R by applying A^{m-i+1} involves one of the vectors x_m, x_1 but not both, and so the relation obtained from R by applying A^{m-i+2} must involve the vector x_2, which is a contradiction. This proves our contention. Since every shortest relation on $\{x_1, x_3, \ldots, x_m\}$ is a full relation, it is clear that any two relations on the latter set are multiples of each other; else, by subtracting an appropriate multiple of one relation from another we would obtain a shorter nonzero relation. This proves that the cone $\operatorname{pos}\{x_1, x_3, \ldots, x_m\}$ is minimal.

Let R denote the relation on $\operatorname{Ext} K$. Since x_2 does not appear in any relation on $\operatorname{Ext} K, x_m, x_1$ must appear on opposite sides of R. So x_1, x_3 also appear on opposite sides of the relation obtained from R by applying A and hence on opposite sides of relation R. Continuing the argument, we infer that for $j = 3, \dots, m-1, x_j$ and x_{j+1} lie on opposite sides of R. It follows that m is odd and R has the same number of terms on its two sides, i.e., R is a balanced relation. ■

It is easy to show the following:

Remark 4.3. For any real numbers p, l with $p \geq 3$, we have

$$(p-1)(l-2) + 2 \leq (p-1)(l-1),$$

where the inequality becomes equality if and only if $p = 3$.

Part (iii) and (iv) of Theorem 4.4, our next result, are not needed in the rest of the paper. They are included for the sake of completeness.

Theorem 4.4. *Let $K \in \mathcal{P}(m, n)$, where $m \geq 4$, and let A be a K-primitive matrix. Then:*

(i) $\gamma(A) \leq (m_A - 1)(m - 1) + 1$, *where the equality holds only if the digraph $(\mathcal{E}, \mathcal{P}(A, K))$ is given by Figure* 1*, in which case $\gamma(A) = (n-1)(m-1) + 1$.*

(ii) $\gamma(A) = (m_A - 1)(m - 1)$ *only if either $(\mathcal{E}, \mathcal{P}(A, K))$ is given by Figure* 1 *or Figure* 2*, in which case $\gamma(A) = (n-1)(m-1)$, or $m_A = 3$.*

(iii) $\gamma(A) = (m_A-1)(m-2)+2$ *only if $m_A \geq 3$ and either $(\mathcal{E}, \mathcal{P}(A, K))$ is given by Figure* 1*, Figure* 2*, Figure* 3*, Figure* 4 *or Figure* 5*, or $(\mathcal{E}, \mathcal{P}(A, K))$ is obtained from Figure* 5 *by deleting any one or two of the three arcs $(\Phi(x_{m-1}), \Phi(x_1))$, $(\Phi(x_m), \Phi(x_1))$ and $(\Phi(x_m), \Phi(x_2))$, or from Figure* 3 *with $m = 4$ by adding the arc $(\Phi(x_3), \Phi(x_1))$ or substituting it for the arc $(\Phi(x_4), \Phi(x_1))$.*

(iv) *If $m_A = 2$ or $(\mathcal{E}, \mathcal{P}(A, K))$ is not given by Figures* 1–5*, nor is derived from Figure* 5 *or from Figure* 3 (*with $m = 4$*) *in the way as described in part* (iii)*, then*

$$\gamma(A) \leq (m_A - 1)(m - 2) + 1.$$

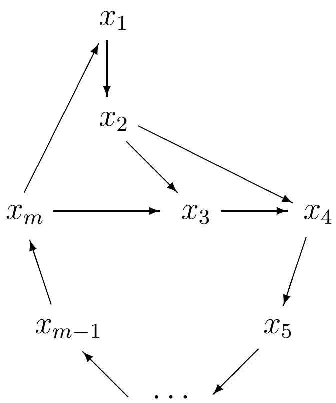

Figure 3.

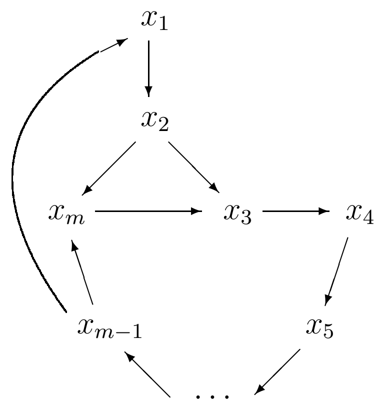

Figure 4.

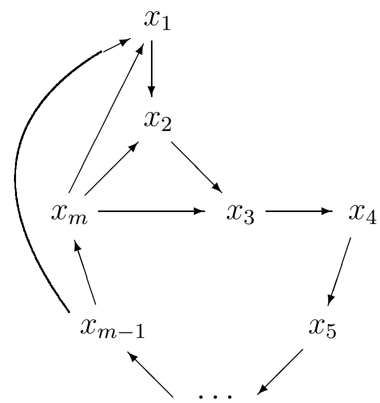

Figure 5.

Proof. When $m_A = 2$, by Lemma 3.6 we have $\gamma(A) \leq 2$. As $m \geq 4$, in this case, the inequality $\gamma(A) \leq (m_A - 1)(m-2) + 1$ is clearly satisfied and none of the equalities $\gamma(A) = (m_A - 1)(m-1)$ or $\gamma(A) = (m_A - 1)(m-2) + 2$ can be attained. Hereafter, we assume that $m_A \geq 3$.

As explained before, the length of the shortest circuit in $(\mathcal{E}, \mathcal{P})$ is at most $m-1$.

(i) Since A is K-primitive, A is non-nilpotent. So the outdegree of each vertex of $(\mathcal{E}, \mathcal{P})$ is positive. Consider any vertex $\Phi(x)$ of the digraph $(\mathcal{E}, \mathcal{P})$. It is clear that $\Phi(x)$ lies on or has access to a circuit of length $l \leq m-1$. By Lemma 3.2 we have

$$\gamma(A, x) \leq (m_A - 2)l + m \leq (m_A - 2)(m-1) + m = (m_A - 1)(m-1) + 1.$$

Since this is true for every nonzero extreme vector x of K, the inequality $\gamma(A) \leq (m_A - 1)(m-1) + 1$ follows.

To establish the desired necessary condition for the inequality to become equality, first we dispense with the case when the length of the shortest circuit in $(\mathcal{E}, \mathcal{P})$ is less than or equal to $m-2$. We contend that in this case every vertex of the digraph lies on or has access to a circuit of length less than or equal to $m-2$. Consider any vertex $\Phi(x)$ of the digraph. As we have explained before, $\Phi(x)$ lies on or has access to some circuit, say $\mathcal{C}$. Choose such a $\mathcal{C}$ of shortest length, say length l. By the definition of l, C contains no chords or loops (unless $\mathcal{C}$ is itself a loop). If $l = m$ then A is not primitive, contradiction. If $l = m-1$ let z be the unique vertex of the digraph not on $\mathcal{C}$. Then, there is an access from $\mathcal{C}$ to z and vice versa, or else A is not primitive. Hence the graph is strongly connected, so $\Phi(x)$ has access to a circuit

of length $m-2$ or less, contradicting $l = m-1$. Hence, necessarily, $l \le m-2$. This proves our contention. So, in this case, we have

$$\gamma(A) \le (m_A - 2)(m-2) + m = (m_A - 1)(m-2) + 2 \le (m_A - 1)(m-1), \quad (4.1)$$

where the first inequality holds by Lemma 3.2 (with $l \le m-2$) and the second inequality follows from Remark 4.3 (with $p = m_A$ and $l = m$).

When the length of the shortest circuit in $(\mathcal{E}, \mathcal{P})$ is $m-1$, by Lemma 4.1 the digraph is given by Figure 1 or Figure 2. If the digraph is given by Figure 2, then each vertex lies on a circuit of length $m-1$ and by Lemma 3.1 we obtain $\gamma(A) \le (m_A - 1)(m-1)$. So when the equality $\gamma(A) = (m_A - 1)(m-1) + 1$ holds, the digraph $(\mathcal{E}, \mathcal{P})$ must be given by Figure 1. In that case, by Lemma 4.2(i), we have $m_A = n$ and hence $\gamma(A) = (n-1)(m-1) + 1$.

(ii) Suppose that the equality $\gamma(A) = (m_A - 1)(m-1)$ holds. The length of the shortest circuit in $(\mathcal{E}, \mathcal{P})$ is either $m-1$ or less. In the former case, by Lemma 4.1 the digraph is given by Figure 1 or Figure 2; then by Lemma 4.2(i), as $m_A = n$ the said equality becomes $\gamma(A) = (n-1)(m-1)$. In the latter case, by the proof of part (i) the inequalities in (4.1) both hold as equality, and by Remark 4.3 we have $m_A = 3$ and hence also $\gamma(A) = 2(m-1)$.

(iii) Suppose that $\gamma(A) = (m_A - 1)(m-2) + 2$. If $(\mathcal{E}, \mathcal{P})$ is given by Figure 1 or Figure 2 we are done, so assume that this is not the case. Note that it is not possible that every vertex of $(\mathcal{E}, \mathcal{P})$ lies on or has access to a circuit of length $\le m-3$ or is at a distance at most 1 to a circuit of length $m-2$, because then by Lemma 3.2 (with $l = m-3$) or by Lemma 3.1 (with $w = 1$ and $l = m-2$) it will follow that $\gamma(A) \le (m_A - 1)(m-2) + 1$. It remains to show that if $(\mathcal{E}, \mathcal{P})$ has a vertex which is at a distance 2 to a circuit of length $m-2$ and which does not lie on or has access to a circuit of length $m-3$ or less, nor is it at a distance at most 1 to another circuit of length $m-2$, then the digraph is given by Figure 3, Figure 4 or Figure 5, or is derived from them in the manner as described in the theorem.

To treat the remaining case we assume that the digraph $(\mathcal{E}, \mathcal{P})$ contains the circuit $\mathcal{C} : \Phi(x_3) \to \Phi(x_4) \to \cdots \to \Phi(x_{m-1}) \to \Phi(x_m) \to \Phi(x_3)$ and also the path $\Phi(x_1) \to \Phi(x_2) \to \Phi(x_3)$. For $i = 2, 3, \ldots, m$, since $\Phi(x_i)$ is at a distance at most 1 to the circuit $\mathcal{C}$, which is of length $m-2$, by Lemma 3.1 we have $\gamma(A, x_i) \le (m_A - 1)(m-2) + 1$. This forces $\gamma(A, x_1) = \gamma(A) = (m_A - 1)(m-2) + 2$, which, in turn, implies that $\Phi(x_1)$ does not lie on or has access to a circuit of length $m-3$ or less, nor is $\Phi(x_1)$ at a distance at most 1 to a circuit of length $m-2$. Therefore, $\mathcal{C}$ does not contain any chords or loops. Besides the arcs on the circuit $\mathcal{C}$ and the

above-mentioned path, $(\mathcal{E}, \mathcal{P})$ certainly has other arcs. We want to find out what possible additional arcs there can be.

Note that there is at least one arc from a vertex of $\mathcal{C}$ to either one of the vertices $\Phi(x_1)$ or $\Phi(x_2)$; else, A^{m-2} maps the extreme ray $\Phi(x_3)$ of K onto itself, which contradicts the K-primitivity of A. Since x_1 is not allowed to lie on a circuit of length $m-2$ or less, none of the arcs $(\Phi(x_j), \Phi(x_1))$, for $j = 2, \ldots, m-2$, can be present. Similarly, since $\Phi(x_1)$ is not allowed to be at a distance 1 to a circuit of length $m-2$ or less, the arcs $(\Phi(x_j), \Phi(x_2))$, for $j = 3, \ldots, m-1$, also cannot be present. So $(\Phi(x_{m-1}), \Phi(x_1)), (\Phi(x_m), \Phi(x_1))$ and $(\Phi(x_m), \Phi(x_2))$ are the only possible arcs from a vertex of $\mathcal{C}$ to either $\Phi(x_1)$ or $\Phi(x_2)$; also, at least one of these three arcs is present.

There cannot exist an arc from $\Phi(x_1)$ to a vertex of $\mathcal{C}$, because in the presence of any such arc the distance from $\Phi(x_1)$ to the circuit $\mathcal{C}$ becomes 1. Similarly, for $m \geq 6$, each of the arcs $(\Phi(x_2), \Phi(x_j)), j = 5, \ldots, m-1$, also cannot exist, because the arc $(\Phi(x_2), \Phi(x_j))$ and one of the arcs $(\Phi(x_{m-1}), \Phi(x_1)), (\Phi(x_m), \Phi(x_1))$ and $(\Phi(x_m), \Phi(x_2))$ (which must be present), together with some of the arcs in $\mathcal{C}$, form either a circuit of length $m-3$ or less, which is at a distance 1 from $\Phi(x_1)$, or a circuit of length $m-2$ or less that contains $\Phi(x_1)$, but this is not allowed. So $(\Phi(x_2), \Phi(x_4))$ and $(\Phi(x_2), \Phi(x_m))$ are the only possible arcs from $\Phi(x_1)$ or $\Phi(x_2)$ to a vertex of $\mathcal{C}$ when $m \geq 6$. The preceding argument does not cover the cases when $m = 4$ or 5. However, we have not ruled out the possibility of the existence of the arcs $(\Phi(x_2), \Phi(x_4))$ and $(\Phi(x_2), \Phi(x_m))$ in these cases.

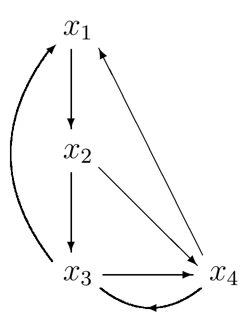

Figure $3'$

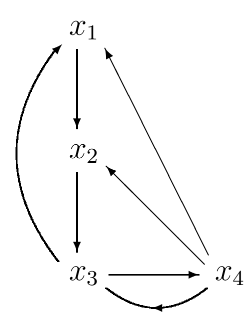

Figure $5'$

Consider the case when the arc $(\Phi(x_2), \Phi(x_4))$ is present. If $m \geq 5$, then the last two of the three arcs $(\Phi(x_m), \Phi(x_1)), (\Phi(x_m), \Phi(x_2))$ and $(\Phi(x_{m-1}), \Phi(x_1))$ cannot be present, else $\Phi(x_1)$ is at a distance at most 1 to a circuit of length $m-2$,

which is not allowed. So in this case the arc $(\Phi(x_m), \Phi(x_1))$ must be present and, furthermore, the arc $(\Phi(x_2), \Phi(x_m))$ also cannot be present (otherwise, we have a circuit of length three containing $\Phi(x_1)$). Therefore, the digraph $(\mathcal{E}, \mathcal{P})$ is given by Figure 3. If $m = 4$, we find that the arc $(\Phi(x_4), \Phi(x_2))$ cannot be present, but the arcs $(\Phi(x_3), \Phi(x_1))$ and $(\Phi(x_4), \Phi(x_1))$ may be present and, indeed, at least one of them must be present. So the digraph $(\mathcal{E}, \mathcal{P})$ is given by Figure $3'$ or is obtained from it by deleting one of the arcs $(\Phi(x_3), \Phi(x_1)), (\Phi(x_4), \Phi(x_1))$. In other words, the digraph is given by Figure 3 (with $m = 4$) or is derived from it in the manner as described in the theorem.

Now suppose that the arc $(\Phi(x_2), \Phi(x_m))$ is present. Using the same kind of argument as before, for $m \geq 5$, one readily rules out the presence of the arcs $(\Phi(x_m), \Phi(x_1))$ and $(\Phi(x_m), \Phi(x_2))$. So in this case the arc $(\Phi(x_{m-1}), \Phi(x_1))$ must be present. Then we can show that the arc $(\Phi(x_2), \Phi(x_4))$ cannot be present. Therefore, the digraph $(\mathcal{E}, \mathcal{P})$ is given by Figure 4. For $m = 4$, we are dealing with the situation when $(\Phi(x_2), \Phi(x_4))$ is an arc, but this has already been treated above (for arbitrary $m \geq 4$).

It remains to consider the case when the arcs $(\Phi(x_2), \Phi(x_4))$ and $(\Phi(x_2), \Phi(x_m))$ are both absent. Then the presence of any one, two or three of the arcs

$$(\Phi(x_{m-1}), \Phi(x_1)), (\Phi(x_m), \Phi(x_1)), (\Phi(x_m), \Phi(x_2))$$

will produce only circuits of length at least $m-1$, but that causes no problem. Then the digraph $(\mathcal{E}, \mathcal{P})$ is given by Figure 5 (which becomes Figure $5'$ when $m = 4$) or is obtained from it by deleting any one or two of the above-mentioned three arcs.

(iv) Now this is obvious. ■

Remark 4.5. Let $K \in \mathcal{P}(3, 3)$, and let A be a K-primitive matrix. Then $\gamma(A) \leq 2m_A - 1$, where the equality holds only if $(\mathcal{E}, \mathcal{P}(A, K))$ is given by Figure 1, in which case $\gamma(A) = 5$.

The preceding remark, in fact, says that part(i) of Theorem 4.4 still holds when $m = n = 3$. It holds by what is known in the 3×3 nonnegative matrix case. However, parts (ii)–(iv) of Theorem 4.4 cannot be extended to the case $m = n = 3$. This is mainly because in that case the equality in part (ii) or (iii) can hold when $m_A = 2$.

By Lemma 4.1, for any polyhedral cone K with $m \geq 3$ extreme rays and any K-primitive matrix A, the length of the shortest circuit in $(\mathcal{E}, \mathcal{P})$ is $m - 2$ or less if

and only if the digraph $(\mathcal{E}, \mathcal{P})$ is not given by Figure 1 or Figure 2 or by a digraph of order 3 whose arc set consists of all possible arcs between every pair of distinct vertices. The proof of Theorem 4.4 (i) shows that in this case $(m_A - 1)(m-2)+2$ is an upper bound for $\gamma(A)$. (The case $m = n = 3$ can be treated separately.) So we have the following

Remark 4.6. For any $K \in \mathcal{P}(m,n)$ and any K-primitive matrix A, if the digraph of $(\mathcal{E}, \mathcal{P}(A,K))$ is not given by Figure 1 or Figure 2 or by a digraph of order 3 whose arc set consists of all possible arcs between every pair of distinct vertices, then $\gamma(A) \leq (n-1)(m-2)+2$.

In below we give another bound for $\gamma(A)$ in terms of m_A, m and s, where s is the length of the shortest circuit in $(\mathcal{E}, \mathcal{P})$. Before we do that, we need to obtain a general result on a digraph first.

Remark 4.7. Let D be a digraph on $m \geq 3$ vertices, each of which has positive out-degree. If the length of the shortest circuit in D is greater than $\lfloor \frac{m-1}{2} \rfloor$, then every vertex of D lies on or has access to a circuit of D of shortest length.

Proof. Since each vertex of D has positive out-degree, each vertex lies on or has access to a circuit. Denote by $s(D)$ the length of the shortest circuit in D. If there is a vertex that does not lie on or has access to a circuit of length $s(D)$, then such vertex must lie on or has access to a circuit, say $\mathcal{C}$, of length $s(D)+1$ or more. It is clear that the circuit $\mathcal{C}$ is vertex disjoint from every circuit of shortest length. Consequently, we have $m \geq s(D)+(s(D)+1)$ or $\lfloor \frac{m-1}{2} \rfloor \geq s(D)$, which is a contradiction. ∎

By Lemma 3.2 and Remark 4.7 we have

Remark 4.8. Let $K \in \mathcal{P}(m,n)$ and let A be a K-primitive matrix. Let s be the length of the shortest circuit in $(\mathcal{E}, \mathcal{P}(A,K))$. If $s > \lfloor \frac{m-1}{2} \rfloor$, then $\gamma(A) \leq s(m_A - 2) + m$.

It is interesting to note that the digraphs given by Figure 3, Figure 4 and Figure 5 are all primitive, like the digraphs given by Figure 1 and Figure 2. Moreover, if A is a K-primitive matrix such that $(\mathcal{E}, \mathcal{P})$ is given by Figure 3, Figure 4 or Figure 5 then A is necessarily nonsingular — this can be proved using the argument given in the proof of Lemma 4.2 (i). However, the digraph obtained from Figure 5′ (i.e., Figure 5 with $m = 4$) by removing the arcs (x_4, x_2) and (x_3, x_1) is strongly connected but not primitive, whereas the one obtained from Figure 3′ (i.e., Figure 3 with $m = 4$)

by removing the arcs (x_4, x_1) and (x_3, x_1) is not even strongly connected. Also, A is singular if it is a K-primitive matrix such that its digraph $(\mathcal{E}, \mathcal{P})$ is derived from Figure 5 by deleting the arcs $(\Phi(x_m), \Phi(x_1))$ and $(\Phi(x_m), \Phi(x_2))$.

Corollary 4.9. *For any $K \in \mathcal{P}(m, n)$ with $m = n + k$, we have $\gamma(K) \leq (n-1)(m-1) + 1 = m^2 - (k+2)m + k + 2$. The equality holds only if there exists a K-primitive matrix A such that the digraph $(\mathcal{E}, \mathcal{P}(A, K))$ is given by Figure 1.*

Proof. Follows from part (i) of Theorem 4.4 (i) and Remark 4.5. ∎

By Corollary 4.9 the answer to Kirkland's conjecture mentioned at the beginning of Section 1 is in the affirmative.

Corollary 4.10. *For any positive integer $m \geq 3$,*

$$\max\{\gamma(K) : K \textit{ is a polyhedral cone with } m \textit{ extreme rays }\} = m^2 - 2m + 2.$$

Proof. Let K be an n-dimensional polyhedral cone with m extreme rays. Since $n \leq m$, by Corollary 4.9, $\gamma(K) \leq (m-1)^2 + 1$. So we have

$$\max\{\gamma(K) : K \text{ is a polyhedral cone with } m \text{ extreme rays}\} \leq m^2 - 2m + 2.$$

On the other hand, by Wielandt's bound we also have $\gamma(\mathbb{R}^m_+) = m^2 - 2m + 2$. Hence, the desired equality follows. ∎

We would like to emphasize that in Corollary 4.10 the number of extreme rays (i.e., m) for the polyhedral cones K under consideration is fixed but there is no restriction on their dimensions (i.e., n).

Hereafter, we call a polyhedral cone $K_0 \in \mathcal{P}(m, n)$ an *exp-maximal cone* if $\gamma(K_0) = \max\{\gamma(K) : K \in \mathcal{P}(m, n)\}$.

5. The minimal cones case

For a non-simplicial polyhedral cone K and a K-primitive matrix A, if the digraph $(\mathcal{E}, \mathcal{P})$ is given by Figure 1 or Figure 2, then Lemma 4.2 gives us some information about K and A. The first lemma of this section shows that if, in addition, K is a minimal cone then the relation for the extreme vectors of K and the action of A on the extreme vectors can be described completely.

Lemma 5.1. *Let $K \in \mathcal{P}(n+1, n), n \geq 3$. Let A be a K-nonnegative matrix.*

(i) *Suppose that $(\mathcal{E}, \mathcal{P}(A, K))$ is given by Figure 1 (with $m = n + 1$, and K is necessarily indecomposable). If n is odd then, after normalization (on the extreme vectors of K and on A), the relation on* $\operatorname{Ext} K$ *and the matrix A are given respectively by* (5.1) *and* (5.2)*:*

$$x_1 + x_3 + \cdots + x_{m-3} + x_{m-1} = x_2 + x_4 + \cdots + x_{m-2} + x_m. \tag{5.1}$$

$$\begin{aligned} Ax_1 &= (1+\alpha)x_2, \\ Ax_i &= x_{i+1} \text{ for } i = 2, 3, \ldots, m-1, \\ Ax_m &= x_1 + \alpha x_2, \end{aligned} \tag{5.2}$$

where α is some positive scalar. If n is even then, after normalization, the relation on $\operatorname{Ext} K$ *and the matrix A are given respectively by* (5.3) *and* (5.4)*:*

$$x_1 + x_2 + x_4 + \cdots + x_{m-3} + x_{m-1} = x_3 + x_5 + \cdots + x_{m-2} + x_m. \tag{5.3}$$

$$\begin{aligned} Ax_1 &= \alpha x_2, \\ Ax_i &= x_{i+1} \text{ for } i = 2, 3, \ldots, m-1, \\ Ax_m &= x_1 + (1+\alpha)x_2, \end{aligned} \tag{5.4}$$

where $\alpha > 0$.

(ii) *Suppose K is indecomposable and $(\mathcal{E}, \mathcal{P}(A, K))$ is given by Figure 2. If n is even then, after normalization, the relation on* $\operatorname{Ext} K$ *and the matrix A are given respectively by* (5.3) *and* (5.5)*, or by* (5.6) *and* (5.7)*:*

$$\begin{aligned} Ax_1 &= \alpha x_2 + (1-\beta)x_3, \\ Ax_2 &= \beta x_3, \\ Ax_i &= x_{i+1}, \text{ for } i = 3, \ldots, m-1, \\ Ax_m &= x_1 + (1+\alpha)x_2, \end{aligned} \tag{5.5}$$

where $\alpha > 0, 0 < \beta < 1$.

$$x_2 + x_3 + x_5 + \cdots + x_{m-2} + x_m = x_1 + x_4 + x_6 + \cdots + x_{m-3} + x_{m-1}. \quad (5.6)$$

$$\begin{aligned} Ax_1 &= (1+\alpha)x_2 + (1+\beta)x_3, \\ Ax_2 &= \beta x_3, \\ Ax_i &= x_{i+1} \text{ for } i = 3, \ldots, m-1, \\ Ax_m &= x_1 + \alpha x_2, \end{aligned} \quad (5.7)$$

where $\alpha, \beta > 0$. *If* n *is odd then, after normalization, the relation on* Ext K *and the matrix* A *are given respectively by* (5.1) *and* (5.8)*:*

$$\begin{aligned} Ax_1 &= (1+\alpha)x_2 + \beta x_3, \\ Ax_2 &= (1+\beta)x_3, \\ Ax_i &= x_{i+1}, i = 3, \ldots, m-1, \\ Ax_m &= x_1 + \alpha x_2, \end{aligned} \quad (5.8)$$

where $\alpha, \beta > 0$.

(iii) *Suppose* K *is decomposable and* $(\mathcal{E}, \mathcal{P}(A, K))$ *is given by Figure* 2. *Then, after normalization, the relation on* Ext K *and the matrix* A *are given respectively by* (5.9) *and* (5.10)*:*

$$x_1 + x_4 + x_6 + \cdots + x_{m-3} + x_{m-1} = x_3 + x_5 + \cdots + x_{m-2} + x_m. \quad (5.9)$$

$$\begin{aligned} Ax_1 &= \alpha x_2 + x_3, \\ Ax_2 &= \beta x_3, \\ Ax_i &= x_{i+1} \text{ for } i = 3, \ldots, m-1, \\ Ax_m &= x_1 + \alpha x_2, \end{aligned} \quad (5.10)$$

where $\alpha, \beta > 0$.

Proof. (i) Since K is non-simplicial and $(\mathcal{E}, \mathcal{P})$ is given by Figure 1, by Lemma 4.2(iii) K is indecomposable. So the (essentially unique) relation on $\operatorname{Ext} K$, which we denote by R, is full. We contend that x_m, x_1 lie on different sides of R. Suppose not. Then x_1, x_2 lie on the same side of the relation obtained from R by applying A. But the latter relation, which is nonzero (as the coefficients of x_1, x_2 are both nonzero), is just a multiple of R, so x_1, x_2 also lie on the same side of relation R. By applying A to the latter relation, we deduce that x_2, x_3 also lie on the same side of relation R. Continuing the argument, we can then show that $x_1, x_2, \ldots, x_m$ all lie on the same side of R, which is impossible, as K is a pointed cone. This proves our contention. The same argument, in fact, also shows that for $j = 2, 3, \ldots, m-1$, x_j, x_{j+1} lie on different sides of R. Now a simple parity count shows that x_2, x_m lie on the same side or opposite sides of R, depending on whether n is odd or even. So when n is odd (i.e., m is even), after normalizing the extreme vectors of K we may assume that relation R is given by (5.1).

As the digraph $(\mathcal{E}, \mathcal{P})$ is given by Figure 1, we have

$$Ax_1 = \beta x_2, Ax_i = \lambda_{i+1} x_{i+1} \text{ for } i = 2, \ldots, m-1 \text{ and } Ax_m = \lambda_1 x_1 + \alpha x_2,$$

where α, β and $\lambda_1, \lambda_3, \lambda_4, \ldots, \lambda_m$ are some positive numbers. Substituting the values of the Ax_i's into the relation obtained from (5.1) by applying A, we obtain the relation:

$$\beta x_2 + \lambda_4 x_4 + \lambda_6 x_6 + \cdots + \lambda_m x_m = \lambda_3 x_3 + \lambda_5 x_5 + \cdots + \lambda_{m-1} x_{m-1} + (\lambda_1 x_1 + \alpha x_2).$$

But relation (5.1) and the above relation are positive multiples of each other, so it follows that we have $\lambda_1 = \lambda_3 = \lambda_4 = \cdots = \lambda_m$ and $\beta = \lambda + \alpha$, where we use λ to denote the common value of the λ_j's. Replacing A by a positive multiple, we may assume that $\lambda = 1$. Then A is given by (5.2).

When n is even, we can show in a similar way that the relation on $\operatorname{Ext} K$ and the matrix A are given by (5.3) and (5.4) respectively.

(ii) We consider the case when n is even first. By the same kind of argument that we have used for part (i) we can show that for $j = 3, \ldots, m$, the vectors x_j, x_{j+1} lie on different sides of the relation on $\operatorname{Ext} K$ (where x_{m+1} is taken to be x_1). Hence, the vectors $x_3, x_5, \ldots, x_m$ lie on one side of the relation and the vectors $x_1, x_4, x_6, \ldots, x_{m-1}$ lie on the other side. As for the vector x_2 it can be on either side. If x_2 is on the same side as x_1 then, after normalizing the extreme vectors of K, we may assume that the relation on $\operatorname{Ext} K$ is given by (5.3); if x_2 lies on the side opposite to x_1, we may assume that the relation is given by (5.6).

We treat the subcase when the relation is given by (5.3), the argument for the remaining subcase being similar. Since the digraph $(\mathcal{E}, \mathcal{P})$ is given by Figure 2, we have

$$Ax_1 = \alpha x_2 + \gamma x_3, Ax_2 = \beta x_3, Ax_i = \lambda_{i+1} x_{i+1} \text{ for } i = 3, \ldots, m-1 \text{ and } Ax_m = \lambda_1 x_1 + \delta x_2,$$

where $\alpha, \beta, \delta, \gamma$ and $\lambda_1, \lambda_4, \ldots, \lambda_m$ are some positive numbers. Applying A to relation (5.3), we obtain the relation:

$$\lambda_4 x_4 + \lambda_6 x_6 + \cdots + \lambda_{m-1} x_{m-1} + (\lambda_1 x_1 + \delta x_2) = (\alpha x_2 + \gamma x_3) + \beta x_3 + \lambda_5 x_5 + \cdots + \lambda_{m-2} x_{m-2} + \lambda_m x_m.$$

But relation (5.3) and the above relation are positive multiples of each other, it follows that we have $\lambda_4 = \lambda_5 = \cdots = \lambda_m = \lambda$, say, and $\lambda_1 = \lambda, \delta = \lambda + \alpha$ and $\gamma + \beta = \lambda$. Replacing A by a positive multiple, we may assume that $\lambda = 1$. Then A is given by equation (5.5).

Now we consider the case when n is odd. Again, we can show that for $j = 3, \ldots, m$, the vectors x_j, x_{j+1} lie on different sides of the relation on $\operatorname{Ext} K$ (where x_{m+1} is taken to be x_1). Hence, $x_1, x_3, x_5, \ldots, x_{m-3}, x_{m-1}$ lie on one side of the unique relation and $x_4, x_6, \ldots, x_{m-2}, x_m$ lie on the other side. If x_1, x_2 lie on the same side of the relation then, since x_1, x_3 also lie on the same side, the same is true for the pair x_2, x_3. Then by applying A we find that x_3, x_4 also lie on the same side of the relation, which contradicts what we have observed above. So x_2 lies on the same side as $x_4, x_6, \ldots, x_m$, and after normalization we may assume that the unique relation is given by (5.1). In a similar way as before we can also show that after normalization A is given by (5.8).

(iii) Suppose K is decomposable and $(\mathcal{E}, \mathcal{P}(A, K))$ is given by Figure 2. By Lemma 4.2(iii), m is odd and K is the direct sum of a ray and an indecomposable minimal cone with a balanced relation for its extreme vectors. The last part of the proof for Lemma 4.2(iii) shows that in the relation on $\operatorname{Ext}(K)$ the vectors $x_1, x_4, x_6, \ldots, x_{m-1}$ lie on one side and the vectors $x_3, x_5, \ldots, x_m$ lie on the other side. After normalizing the extreme vectors of K, we may assume that the relation on $\operatorname{Ext} K$ is given by relation (5.9). By the same kind of argument as before, we can also show that A, after normalization, is given by equation (5.10). ∎

Lemma 5.2. *Let $K \in \mathcal{P}(m, n)$ be a minimal cone with extreme vectors $x_1, \ldots, x_m$ (where $m = n + 1$), and let A be a K-nonnegative matrix. Let the relations* (5.1), (5.3), (5.6), (5.9) *on the extreme vectors of K and the equations* (5.2), (5.4), (5.5), (5.7), (5.8), (5.10) *on the action of A be as given in Lemma 5.1. Then*:

(i) $(\mathcal{E}, \mathcal{P}(A, K))$ *is given by Figure* 1 (*and* K *is indecomposable*) *if and only if after normalization the relation on* $\operatorname{Ext} K$ *and the matrix* A *are given respectively by relation* (5.1) *and equation* (5.2), *in which case* n *is odd and* $\gamma(A) = n^2 - n + 1$, *or by relation* (5.3) *and equation* (5.4), *in which case* n *is even and* $\gamma(A) = n^2 - n$.

(ii) $(\mathcal{E}, \mathcal{P}(A, K))$ *is given by Figure* 2 *and* K *is indecomposable if and only if after normalization the relation on* $\operatorname{Ext} K$ *and the matrix* A *are given respectively by relation* (5.6) *and equation* (5.7), *in which case* n *is even and* $\gamma(A) = n^2 - n$, *or by relation* (5.3) *and equation* (5.5), *in which case* n *is even and* $\gamma(A) = n^2 - n - 1$, *or by relation* (5.1) *and equation* (5.8), *in which case* n *is odd and* $\gamma(A) = n^2 - n$.

(iii) $(\mathcal{E}, \mathcal{P}(A, K))$ *is given by Figure* 2 *and* K *is decomposable if and only if after normalization the relation on* $\operatorname{Ext} K$ *and the matrix* A *are given respectively by relation* (5.9) *and equation* (5.10), *in which case* n *is even and* $\gamma(A) = n^2 - n$.

Proof. The "only if" parts of (i), (ii) and (iii) are done in Lemma 5.1. It remains to treat the "if" parts and the parts concerning the value of $\gamma(A)$.

(i) First, suppose the relation on $\operatorname{Ext} K$ and the matrix A are given respectively by relation (5.1) and equation (5.2). It is clear that m is even, and so n is odd. Note that A is well-defined, as it preserves the relation on $\operatorname{Ext} K$. We contend that the digraph $(\mathcal{E}, \mathcal{P})$ is given by Figure 1. It is clear that for $i = 1, \dots, m-1, (\Phi(x_i), \Phi(x_{i+1}))$ is the only outgoing arc from vertex $\Phi(x_i)$. By definition $Ax_m = x_1 + \alpha x_2$, so $\Phi(Ax_m)$ equals $\Phi(x_1 + x_2)$, which is the smallest face of K containing x_1, x_2. Since relation (5.1), the (unique) relation on the extreme vectors of K, is full, K is indecomposable. As x_1, x_2 lie on opposite sides of (5.1), by Theorem A, x_1, x_2 lie on a common maximal face of K, i.e., $\Phi(x_1 + x_2)$ is a nontrivial face. But every nontrivial face of an indecomposable minimal cone is simplicial, so x_1, x_2 are the only extreme vectors of $\Phi(x_1 + x_2)$. It follows that $(\Phi(x_m), \Phi(x_1))$ and $(\Phi(x_m), \Phi(x_2))$ are the only outgoing arcs from vertex $\Phi(x_m)$. This proves our contention. Now a straightforward calculation yields the following: $A^{m-1}x_1 = (1+\alpha)x_m$; $A^m x_1 = (1+\alpha)(x_1 + \alpha x_2)$, i.e., $\Phi(A^m x_1) = \Phi(x_1 + x_2)$; and $\Phi(A^{j(m-1)}x_1) = \Phi(x_{m-j+1} + x_{m-j+2} + \cdots + x_{m-1} + x_m)$ for $j = 1, \dots, m-2$. So $A^{(n-1)(m-1)}x_1$ is a positive linear combination of $x_3, x_4, \dots, x_m$ and by Theorem A it belongs to the relative interior of a maximal face of K. On the other hand, $A^{(n-1)(m-1)+1}x_1$ belongs to $\operatorname{int} K$ as it can be written as a positive linear combination of all x_i except x_3. Thus

$\gamma(A, x_1) = (n-1)(m-1)+1 = n^2-n+1$. But $(\mathcal{E}, \mathcal{P})$ is given by Figure 1, so by Lemma 4.2(ii), $\gamma(A) = \gamma(A, x_1) = n^2-n+1$.

Next, suppose that the relation on $\operatorname{Ext} K$ and the matrix A are given by relation (5.3) and equation (5.4) respectively. Then K is indecomposable. Note the left side of relation (5.3) has one term more than its right side and it contains both x_1, x_2. Since $m(\geq 4)$ is odd, the left side has at least three terms; so $Ax_m = x_1 + (1+\alpha)x_2 \in \partial K$. As before one can verify that $(\mathcal{E}, \mathcal{P})$ is given by Figure 1. Also, a straightforward calculation shows that $A^{(n-1)(m-1)-1}x_1$, being a positive linear combination of $x_2, x_3, \ldots, x_{m-1}$, belongs to ∂K, whereas $A^{(n-1)(m-1)}x_1$, being a positive linear combination of $x_3, x_4, \ldots, x_m$, belongs to $\operatorname{int} K$. Since $(\mathcal{E}, \mathcal{P})$ is given by Figure 1, we have $\gamma(A) = \gamma(A, x_1) = (n-1)(m-1) = n^2-n$.

(ii) Suppose the relation on $\operatorname{Ext} K$ and the matrix A are given by relation (5.6) and equation (5.7) respectively. Since relation (5.6) is full, K is indecomposable. Note that the left side of (5.6) has at least three terms and it contains both x_2 and x_3, so Ax_1, which is a positive linear combination of x_2, x_3, belongs to ∂K. Similarly, $Ax_m (= x_1 + \alpha x_2)$ also belongs to ∂K, as x_1, x_2 lie on opposite sides of (5.6). By the same kind of argument as given in the proof for part (i), one readily verifies that $(\mathcal{E}, \mathcal{P})$ is given by Figure 2. Also, $A^{(n-1)(m-1)-1}x_2$, being a positive linear combination of $x_3, x_4, \ldots, x_m$, belongs to ∂K, whereas $A^{(n-1)(m-1)}x_2$ belongs to $\operatorname{int} K$, as it can be written as a positive linear combination of all the x_i's except x_3. Since $(\mathcal{E}, \mathcal{P})$ is given by Figure 2, we have $\gamma(A) = \gamma(A, x_2) = (n-1)(m-1) = n^2-n$.

When the relation on $\operatorname{Ext} K$ and the matrix A are given by relation (5.3) and equation (5.5) respectively, we can show that K is indecomposable and also that $(\mathcal{E}, \mathcal{P})$ is given by Figure 2. In this case, $A^{(n-1)(m-1)-2}x_2$, being a positive linear combination of $x_2, x_3, \ldots, x_{m-1}$, belongs to ∂K, whereas $A^{(n-1)(m-1)-1}x_2$, being a positive linear combination of $x_3, x_4, \ldots, x_m$, belongs to $\operatorname{int} K$. It follows that we have $\gamma(A) = \gamma(A, x_2) = (n-1)(m-1)-1 = n^2-n-1$.

Similarly, we can show that when the relation and the matrix A are given by relation (5.1) and equation (5.8) respectively, K is indecomposable, $(\mathcal{E}, \mathcal{P})$ is given by Figure 2 and $\gamma(A) = n^2-n$.

(iii) Suppose the relation on $\operatorname{Ext} K$ and the matrix A are given by relation (5.9) and equation (5.10) respectively. In this case $K = \operatorname{pos}\{x_2\} \oplus \operatorname{pos}\{x_1, x_j, 3 \leq j \leq m\}$. We readily check that $(\mathcal{E}, \mathcal{P})$ is given by Figure 2. A straightforward calculation shows that $A^{(m-1)(m-2)-1}x_2$ is a positive linear combination of $x_3, x_4, \ldots, x_m$. So $A^{(n-1)(m-1)-1}x_2$ belongs to the indecomposable summand $\operatorname{pos}\{x_1, x_j, 3 \leq j \leq m\}$ of K and hence lies in ∂K. On the other hand, $A^{(n-1)(m-1)}x_2$ belongs to $\operatorname{int} K$, as it can be written as a positive linear combination of all x_i's except x_3. Hence we have

$\gamma(A) = \gamma(A, x_2) = (n-1)(m-1) = n(n-1)$. ■

Theorem 5.3. *Let $n \geq 3$ be a given positive integer.*

(I) *The quantity* $\max\{\gamma(K) : K \in \mathcal{P}(n+1, n)\}$ *equals $n^2 - n + 1$ if n is odd and equals $n^2 - n$ if n is even.*

(II) *Suppose n is odd.*

   (i) *An n-dimensional minimal cone is exp-maximal if and only if the cone is indecomposable and the relation for its extreme vectors is balanced.*

   (ii) *Let K be an n-dimensional exp-maximal minimal cone. A K-primitive matrix A is exp-maximal if and only if the digraph $(\mathcal{E}, \mathcal{P}(A, K))$ is given by Figure 1.*

(III) *Suppose n is even.*

   (i) *An n-dimensional minimal cone is exp-maximal if and only if either the cone is indecomposable and has a balanced relation for its extreme vectors, or it is the direct sum of a ray and an indecomposable minimal cone with a balanced relation for its extreme vectors.*

   (ii) *Let K be an indecomposable exp-maximal minimal cone. A K-primitive matrix A is exp-maximal if and only if the digraph $(\mathcal{E}, \mathcal{P}(A, K))$ is, upon relabelling its vertices suitably, given by Figure 1 or Figure 2, and in the latter case x_1, x_2 are required to appear on opposite sides of the relation for the extreme vectors of K.*

   (iii) *Let K be a decomposable exp-maximal minimal cone. A K-primitive matrix A is exp-maximal if and only if the digraph $(\mathcal{E}, \mathcal{P}(A, K))$ is given by Figure 2.*

Proof. We first observe that when n is even, there is no minimal cone K such that $\gamma(K) = n^2 - n + 1$. Assume to the contrary that there is one such K. Choose a K-primitive matrix A that satisfies $\gamma(A) = n^2 - n + 1$. By Corollary 4.9 $(\mathcal{E}, \mathcal{P})$ is given by Figure 1. Since n is even, by the second half of Lemma 5.1(i), after normalization, the relation on $\operatorname{Ext} K$ and the matrix A are given by relation (5.3) and equation (5.4) respectively. So by Lemma 5.2(i), we have $\gamma(A) = n^2 - n$, which is a contradiction.

For any positive integer n, by Corollary 4.9, $\gamma(K) \leq n^2 - n + 1$ for every n-dimensional minimal cone K.

Let n be odd. Take any n-dimensional indecomposable cone K for which the relation on its extreme vectors has the same number of terms on its two sides. After re-indexing and normalizing the extreme vectors $x_1, \dots, x_m$ of K, we may assume that the relation on $\mathrm{Ext}\, K$ is given by (5.1). Let A be the $n \times n$ real matrix given by (5.2). By Lemma 5.2(i) A is K-primitive and $\gamma(A) = n^2 - n + 1$. So we have $\gamma(K) = n^2 - n + 1$. This establishes (I) for odd n as well as the "if" part of (II)(i).

Now let n be even. In view of the above observations, the maximum value of $\gamma(K)$ as K runs through all n-dimensional minimal cones is at most $n^2 - n$. We are going to show that the value $n^2 - n$ can be attained.

Take any indecomposable minimal cone K such that in the relation on $\mathrm{Ext}\, K$ the number of vectors on its two sides differs by 1. Normalizing the extreme vectors of K, we may assume that the relation is given by (5.3). Let A be the matrix given by (5.4). By Lemma 5.2(i) we have $\gamma(A) = n^2 - n$. For this K, certainly we have $\gamma(K) = n^2 - n$. This establishes (I) for even n and completes the proof for (I).

If K is the direct sum of a ray and an indecomposable minimal cone for which the relation on its extreme vectors has same number of terms on its two sides, then necessarily n is even and after normalization we may assume that the relation is given by (5.9). By Lemma 5.2(iii), the matrix A defined by (5.10) satisfies $\gamma(A) = n^2 - n$.

So we have also established the "if" part of (III)(i).

To prove the "only if" part of (II)(i), assume that n is odd and let K be an n-dimensional minimal cone that satisfies $\gamma(K) = n^2 - n + 1$. By Corollary 4.9 there exists a K-primitive matrix A such that the digraph $(\mathcal{E}, \mathcal{P})$ is given by Figure 1. By Lemma 4.2(iii), K is indecomposable. By Lemma 5.1(i), after normalizing the extreme vectors of K, we may assume that the relation on $\mathrm{Ext}\, K$ is given by relation (5.1). So the relation has the same number of terms on its two sides.

The "only if" part of (II)(ii) follows from part(i) of Theorem 4.4 (by taking $m = n + 1$ and $m_A = n$), whereas its "if" part is a consequence of Lemma 5.2(i).

The proof of part(II) is complete.

To prove the "only if" part of (III)(i), assume that n is even and let K be an n-dimensional minimal cone such that $\gamma(K) = n^2 - n$. Choose a K-primitive matrix A such that $\gamma(A) = \gamma(K)$. By part (ii) of Theorem 4.4, in this case we have $m_A = n$ and either $(\mathcal{E}, \mathcal{P})$ is given by Figure 1 or Figure 2, or $m_A = 3$. The case $m_A = 3$ cannot happen; otherwise, $n = 3$, contradicting the assumption that n is even. Then, by Lemma 4.2(iii), K is either indecomposable or is the direct sum of a ray and an indecomposable minimal cone for which the relation on its extreme

vectors has the same number of terms on its two sides. In the latter case, we are done. In the former case, $(\mathcal{E}, \mathcal{P})$ is given by Figure 1 or Figure 2. If the digraph is given by Figure 1 then, since K is indecomposable minimal, by Lemma 5.1(i), after normalization, the relation on $\operatorname{Ext} K$ is given by (5.3). If the digraph is given by Figure 2, then by part (ii) of the same lemma, after normalization, the relation on $\operatorname{Ext} K$ and the matrix A are given respectively by (5.3) and (5.5) or (5.6) and (5.7). (We have to rule out the former possibility, because by Lemma 5.2(ii) we have $\gamma(A) = n^2 - n - 1$, which is a contradiction.) In any case, in the relation on $\operatorname{Ext} K$ the number of terms on its two sides differ by 1.

To prove the "only if" part of (III)(ii), let K be an indecomposable minimal cone such that in the relation on its extreme vectors the number of terms on its two sides differ by 1, and suppose that A is a K-primitive matrix such that $\gamma(A) = n^2 - n$. By the above proof for the "only if" part of (III)(i), $(\mathcal{E}, \mathcal{P})$ is given by Figure 1 or Figure 2. If it is given by Figure 2 then after normalization the relation on $\operatorname{Ext} K$ is given by relation (5.6), hence x_1, x_2 lie on opposite sides of the relation.

To prove the "if" part of (III)(ii), first suppose that $(\mathcal{E}, \mathcal{P})$ is given by Figure 1. Since n is even, by Lemma 5.2(i), after normalization, the relation on $\operatorname{Ext} K$ and the matrix A are given respectively by (5.3) and (5.4), and we have $\gamma(A) = n^2 - n$. If the digraph is given by Figure 2 and x_1, x_2 appear on opposite sides of the relation on $\operatorname{Ext} K$, then since K is indecomposable and n is even, by Lemma 5.2(ii) after normalization the relation on $\operatorname{Ext} K$ and the matrix A are given respectively by (5.6) and (5.7), and we have $\gamma(A) = n^2 - n$.

The "if" part of (III)(iii) follows from Lemma 5.2(iii). To prove its "only if" part, let A be a K-primitive matrix such that $\gamma(A) = n^2 - n$. As explained in the above proof for the "only if" part of (III)(i), the digraph $(\mathcal{E}, \mathcal{P})$ is given by either Figure 1 or Figure 2. By Lemma 4.2(iii), if it is given by Figure 1 then K is necessarily indecomposable. But now K is decomposable, so the digraph is given by Figure 2.

The proof is complete. ■

By Corollary 4.9 and Theorem 5.3(I) we readily deduce the following result, which is an improvement of the already-proved Kirkland's conjecture.

Corollary 5.4. *For any positive integer $m \geq 4$, the maximum value of $\gamma(K)$ as K runs through non-simplicial polyhedral cones with m extreme rays and of all possible dimensions is $m^2 - 3m + 3$ if m is even, and is $m^2 - 3m + 2$ if m is odd.*

6. The 3-dimensional case

Two distinct extreme rays $\Phi(x), \Phi(y)$ (or, distinct extreme vectors x, y) of K are said to be *neighborly* if $x + y \in \partial K$.

Lemma 6.1. *Let K be a 3-dimensional polyhedral cone with extreme vectors $x_1, \ldots, x_m$, where $m \geq 3$. Let $A \in \pi(K)$ and suppose that $(\mathcal{E}, \mathcal{P}(A, K))$ is given by Figure 1 or Figure 2. Then:*

(i) *For $i = 1, \ldots, m, \Phi(x_i)$ and $\Phi(x_{i+1})$ (where $\Phi(x_{m+1})$ is taken to be $\Phi(x_1)$) are neighborly extreme rays of K.*

(ii) *$\gamma(A)$ equals $2m - 1$ or $2m - 2$ depending on whether $(\mathcal{E}, \mathcal{P}(A, K))$ is given by Figure 1 or by Figure 2.*

Note that for $i = 1, \ldots, m, \Phi(x_i)$ and $\Phi(x_{i+1})$ are adjacent vertices of the digraph $(\mathcal{E}, \mathcal{P})$ (when it is given by Figure 1 or Figure 2). However, it is not clear that $\Phi(x_i)$ and $\Phi(x_{i+1})$ are neighborly extreme rays of the cone K.

Proof of Lemma 6.1. First, we consider the case when the digraph $(\mathcal{E}, \mathcal{P})$ is given by Figure 1. It is not difficult to establish the following:

Assertion. Let C be a convex polygon in $\mathbb{R}^2$ with extreme points $w_1, \ldots, w_m$ and edges $\overline{w_i w_{i+1}}, i = 1, \ldots, m$, where w_{m+1} is taken to be w_1 and $\overline{w_i w_{i+1}}$ denotes the line segment joining the points w_i and w_{i+1}. Let $\tilde{C}$ be the polygon with extreme points $w_1', w_2, w_3, \ldots, w_m$, where $w_1' = (1 - \lambda)w_1 + \lambda w_2$ for some $\lambda, 0 < \lambda < 1$. Then the edges of $\tilde{C}$ are $\overline{w_1' w_2}, \overline{w_i w_{i+1}}, i = 2, \ldots, m - 1$ and $\overline{w_m w_1'}$.
The fact that $(\mathcal{E}, \mathcal{P})$ is given by Figure 1 implies that Ax_i is a positive multiple of x_{i+1} for $i = 1, \ldots, m-1$ and, moreover, $\Phi(Ax_m)$ is a 2-dimensional face of K with extreme rays $\Phi(x_1)$ and $\Phi(x_2)$. Hence, $\Phi(x_1), \Phi(x_2)$ are neighborly extreme rays of K and we have $Ax_m = \alpha_1 x_1 + \alpha_2 x_2$ for some positive numbers α_1, α_2. So AK is generated by the (pairwise distinct) extreme vectors $x_1', x_2, x_3, \ldots, x_m$, where $x_1' := Ax_m$. Using an equivalent formulation of the above assertion in terms of 3-dimensional polyhedral cones, we see that for all $i, j, 2 \leq i, j \leq m, x_i, x_j$ are neighborly extreme vectors of AK if and only if they are neighborly extreme vectors of K. (Note that here we are not using (i), something that we have not yet established.)

By Lemma 4.2(i) A is nonsingular; so the cones K and AK are linearly isomorphic under A. Since x_1, x_2 are neighborly extreme vectors of K, Ax_1, Ax_2 are

neighborly extreme vectors of AK. But Ax_1, Ax_2 are respectively positive multiples of x_2 and x_3, so x_2, x_3 are neighborly extreme vectors of AK and, in view of what we have done above, x_2, x_3 are also neighborly extreme vectors of K. By repeating the argument, we can show that for $i = 2, 3, \ldots, m-1$, x_i and x_{i+1} (also, x_i and x_{i-1}) are neighborly extreme vectors of K. It is clear that the remaining extreme vector x_1 is neighborly to x_m (and x_2).

By direct calculation, $A^{2(m-1)}x_1$ is a positive linear combination of x_{m-1} and x_m, and as x_{m-1} and x_m are neighborly extreme vectors, $A^{2(m-1)}x_1 \in \partial K$. On the other hand, $A^{2m-1}x_1$ is a positive linear combination of x_m, x_1 and x_2, so it belongs to $\operatorname{int} K$. This shows that $\gamma(A, x_1) = 2m - 1$. In view of Lemma 4.2(ii), we have $\gamma(A) = \gamma(A, x_1) = 2m - 1$.

When the digraph $(\mathcal{E}, \mathcal{P})$ is given by Figure 2, we employ a similar argument. For convenience, we denote the extreme vectors of AK by $y_1, \ldots, y_m$, where $y_1 = Ax_m, y_2 = Ax_1$ and $y_i = x_i$ for $i = 3, \ldots, m$. In this case, y_1 (respectively, y_2) is a positive linear combination of x_1 and x_2 (respectively, x_2 and x_3), and the extreme vectors x_2, x_1 of K are neighborly, and so are x_2 and x_3. Moreover, K and AK are still linearly isomorphic under A. Using an assertion similar to the one given above, we can show that for $i, j = 3, \ldots, m, y_i, y_j$ are neighborly extreme vectors of AK if and only if x_i, x_j are neighborly extreme vectors of K. Inductively we can show that x_i, x_{i+1} (also, x_i and x_{i-1}) are neighborly extreme vectors of K for $i = 3, \ldots, m-1$. Also, we can conclude that the remaining extreme vector x_1 of K is neighborly to x_m (and x_2).

By direct calculation, $A^{2m-3}x_2$ is a positive linear combination of x_{m-1} and x_m, whereas $A^{2m-2}x_2$ is a positive linear combination of x_m, x_1 and x_2; so $A^{2m-3}x_2 \in \partial K$ and $A^{2m-2}x_2 \in \operatorname{int} K$. By Lemma 4.2(ii) we have $\gamma(A) = \gamma(A, x_2) = 2m - 2$. ■

According to Lemma 4.2(i), for any $K \in \mathcal{P}(m, n), A \in \pi(K)$, if $(\mathcal{E}, \mathcal{P}(A, K))$ is given by Figure 1 or Figure 2, then A has an annihilating polynomial of the form $t^m - ct - d$, where $c, d > 0$. Note that if, in addition, $\rho(A) = 1$ then necessarily $c + d = 1$. Conversely, if A is K-nonnegative and has an annihilating polynomial of the form $t^m - ct - (1 - c)$, where $0 < c < 1$, then necessarily $\rho(A) = 1$. This follows from an application of the Perron-Frobenius theory to A, because then 1 is the only positive real root of the polynomial, in view of Descartes' rule of signs, which says that the number of positive roots of a polynomial either is equal to the number of its variations of sign or is less than that number by an even integer, a root of multiplicity k being counted as k roots. (The fact that 1 is the only positive real root of the polynomial can also be shown directly by proper factorization.)

We will see that in constructing examples of K and A such that $(\mathcal{E}, \mathcal{P}(A, K))$ is given by Figure 1, polynomials of the form $t^m - ct - (1-c)$, where $0 < c < 1$, play a role. In our next result, we study the roots of a polynomial of the said form.

Lemma 6.2. *Consider the polynomial*

$$h(t) = t^m - ct - (1-c),$$

where $m \geq 3$ and $c \in (0,1)$.

(i) *The roots of $h(t)$ are all simple unless m is odd and $c = c_m$, where c_m denotes the unique real root in $(0,1)$ of the equation*

$$\frac{(m-1)^{m-1}}{m^m} t^m = (t-1)^{m-1}.$$

(ii) *When m is even, $h(t)$ has precisely one real root other than 1. When m is odd, besides the root 1, $h(t)$ has precisely two real roots if $c > c_m$, one real root (which is a double root) if $c = c_m$ and no real roots if $c < c_m$. In all cases, each real root of $h(t)$ other than 1 lies in $(-1, 0)$.*

(iii) *When $m \geq 4$ or when $m = 3$ and $c < \frac{3}{4}(= \mathrm{c}_3)$, the polynomial $h(t)$ has a unique complex root of the form $re^{i\theta}$, where $r > 0$ and $\theta \in (\frac{2\pi}{m}, \frac{2\pi}{m-1})$. Moreover, r and θ are related in that r is the unique positive real root, which is less than 1, of the polynomial $g_\theta(t)$ given by:*

$$g_\theta(t) = \frac{\sin(m-1)\theta}{\sin\theta} t^m - \frac{\sin m\theta}{\sin\theta} t^{m-1} + 1.$$

Three remarks are in order. First, by Descartes' rule of signs, one can show that the polynomial $h(t)$, considered in the lemma, has exactly one positive real root, and also that it has exactly one negative real root if m is even and either two (counting multiplicities) or no negative real roots if m is odd. Moreover, since the coefficients of the polynomial $h(-t-1)$ are all positive if m is even, and all negative if m is odd, the polynomial $h(t)$ always has no real root less than -1. Of course, this agrees with part (ii) of the lemma, but the lemma contains more information. Second, actually every root of $h(t)$ other than 1 has modulus strictly less than 1. Here is a one-line proof: If λ is a root of $h(t)$, then $|\lambda|^m = |c\lambda + (1-c)| \leq c|\lambda| + (1-c)$, which is possible only if $\lambda = 1$ or $|\lambda| < 1$. Third, part (iii) of the lemma follows from

properly combining Theorem 2 in [18] and Lemma 3 in [17], at least for $m \geq 4$. (The assumptions made in [18], namely (2.2) there, seems to rule out the case $m = 3$ of our theorem. In his notation we have $d = n, k = 1$ and $s = 1$ and his n is our m.) We give a proof for the sake of completeness.

Proof of Lemma 6.2. (i) Clearly, the roots of $h'(t)$ are precisely all the $(m-1)$th roots of $\frac{c}{m}$. A little calculation shows that if t_0 is a common root of $h(t)$ and $h'(t)$ then $\frac{ct_0}{m} = t_0^m = ct_0 + (1-c)$ and so $ct_0(\frac{1}{m} - 1) = 1 - c$, which implies that t_0 is the negative $(m-1)$th real root of $\frac{c}{m}$ and m is odd. So $h(t)$ and $h'(t)$ have a common root if and only if m is odd and the negative $(m-1)$th real root of $\frac{c}{m}$ is a root of $h(t)$. By calculation one finds that the latter condition, in turn, is equivalent to the condition that m is odd and c is a root of the polynomial $\alpha_m t^m - (t-1)^{m-1}$, where $\alpha_m = \frac{(m-1)^{m-1}}{m^m}$. When m is odd, by considering the said polynomial and its derivative we readily show that the polynomial has a unique real root in $(0, 1)$, which we denote by c_m. So we can conclude that the roots of $h(t)$ are all simple unless m is odd and $c = c_m$.

(ii) Rewriting $h(t)$, we have, $h(t) = (t^m - 1) - c(t-1) = (t-1)(p(t) - c)$, where $p(t) = t^{m-1} + t^{m-2} + \cdots + t + 1$. So 1 is always a root of $h(t)$ and for any complex number $w \neq 1$, w is a root of $h(t)$ if and only if w is a root of the equation $p(t) = c$.

When m is even, a consideration of the derivative of $p(t)$ shows that $p(t)$ is a strictly increasing function on the real line. But $p(-1) = 0, p(0) = 1$ and $c \in (0, 1)$, so the equation $p(t) = c$ has exactly one real root and that real root belongs to $(-1, 0)$. Hence $h(t)$ has precisely one real root other than 1 and this root lies in $(-1, 0)$.

When m is odd, on the real line $p(t)$ is a strictly convex function, since its second derivative always takes positive values, as can be shown by some calculation. It is straightforward to show that a complex number z_0 is a common root of $h(t)$ and $h'(t)$ if and only if $p(z_0) = c$ and $p'(z_0) = 0$. On the other hand, by part (i) and its proof, $h(t)$ and $h'(t)$ have a common root if and only if $c = c_m$; in that case, the common root is unique and is equal to the negative $(m-1)$th real root of $\frac{c_m}{m}$. So c_m is, in fact, the absolute minimum value of $p(t)$. Hence, the equation $p(t) = c$ has two distinct real roots if $c > c_m$, one real root (which is a double root) if $c = c_m$, and no real roots if $c < c_m$. As $p(-1) = p(0) = 1$ and $p(t)$ is strictly convex, each real root of the equation $p(t) = c$ or, in other words, each real root of $h(t)$ other than 1, must belong to $(-1, 0)$.

(iii) First, we establish the uniqueness of the root of $h(t)$ in the desired polar form.

Let $z_1 = r_1 e^{i\theta_1}, z_2 = r_2 e^{i\theta_2}$, where $r_1, r_2 > 0$ and $\theta_1, \theta_2 \in (\frac{2\pi}{m}, \frac{2\pi}{m-1})$, be two different roots of $h(t)$. Then from $z_1^m - cz_1 - (1-c) = 0$ and $z_2^m - cz_2 - (1-c) = 0$ we obtain $(z_1 - z_2)(\sum_{k=0}^{m-1} z_1^{m-1-k} z_2^k) = c(z_1 - z_2)$, which implies that $c = \sum_{k=0}^{m-1} z_1^{m-1-k} z_2^k$. For any nonzero complex number z, denote by $\arg(z)$ the argument of z that belongs to $(0, 2\pi]$. Since $\theta_1, \theta_2 \in (\frac{2\pi}{m}, \frac{2\pi}{m-1})$, we have $\arg(z_1^{m-1-k} z_2^k) \in (\frac{2\pi(m-1)}{m}, 2\pi)$ for $k = 0, \ldots, m-1$. Thus, for each k, the complex number $z_1^{m-1-k} z_2^k$ belongs to the relative interior of the convex cone in the complex plane generated by 1 and $e^{-\frac{2\pi}{m} i}$, and hence so does the sum $\sum_{k=0}^{m-1} z_1^{m-1-k} z_2^k$, which is a contradiction as c is a positive real number.

Next, we contend that for any real number $\theta \in (\frac{2\pi}{m}, \frac{2\pi}{m-1})$, the polynomial $g_\theta(t)$ has a unique positive real root and this root is less than 1.

We have $g_\theta(0) = 1 > 0$ and

$$g_\theta(1) = \frac{\sin(m-1)\theta - \sin m\theta}{\sin\theta} + 1 = -\frac{\cos(m - \frac{1}{2})\theta}{\cos\frac{\theta}{2}} + 1 < 0,$$

where the second equality follows from the trigonometric identity $\sin\alpha - \sin\beta = 2\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$ and the inequality holds as $2\pi - \frac{1}{2}\theta < (m - \frac{1}{2})\theta < 2\pi + \frac{\theta}{2}$ and $0 < \frac{\theta}{2} < \frac{\pi}{2}$. In addition, we also have

$$g_\theta'(t) = \frac{t^{m-2}}{\sin\theta}\left[mt\sin(m-1)\theta - (m-1)\sin m\theta\right] < 0$$

for all $t \in (0, \infty)$, as $\sin m\theta > 0$ and $\sin(m-1)\theta < 0$. So it is clear that the polynomial $g_\theta(t)$ has a unique positive real root and this root is less than 1. (The latter assertion can also be established by using Descartes' rule of signs.)

Now define a real-valued function ζ on $(\frac{2\pi}{m}, \frac{2\pi}{m-1})$ by $\zeta(\theta) = r_\theta^{m-1}\frac{\sin m\theta}{\sin\theta}$, where r_θ denotes the unique positive real root of the polynomial $g_\theta(t)$. Note that ζ is a continuous function, as r_θ depends on θ continuously. Also, we have $0 < \zeta(\theta) < 1$, as $2\pi < m\theta < 2\pi + \theta$ and $0 < r_\theta < 1$.

It is readily checked that a complex number $re^{i\theta}$ (in polar form) is a root of the polynomial $h(t)$ if and only if we have

$$(1-c) + cr\cos\theta = r^m \cos m\theta, \tag{6.1}$$

$$cr\sin\theta = r^m \sin m\theta. \tag{6.2}$$

Rewriting (6.2) and adding $\cos\theta$ times (6.2) to $-\sin\theta$ times (6.1) (and noting that $\sin\theta, \cos\theta \neq 0$ for $\theta \in (\frac{2\pi}{m}, \frac{2\pi}{m-1})$), we find that (6.1) and (6.2) holds if and only if

$$c = r^{m-1}\frac{\sin m\theta}{\sin\theta} = \frac{r^m \sin(m-1)\theta}{\sin\theta} + 1.$$

So, for any $\theta \in (\frac{2\pi}{m}, \frac{2\pi}{m-1})$, $r_\theta e^{i\theta}$ is a root of $h(t)$ if $c = \zeta(\theta)$. To complete our proof, it remains to show that the function ζ maps the open interval $(\frac{2\pi}{m}, \frac{2\pi}{m-1})$ onto $(0,1)$ when $m \geq 4$ and onto $(0, \frac{3}{4})$ when $m = 3$.

Note that the function ζ is one-to-one. Otherwise, there exist $\theta_1, \theta_2 \in (\frac{2\pi}{m}, \frac{2\pi}{m-1})$, $\theta_1 \neq \theta_2$, such that $\zeta(\theta_1) = \zeta(\theta_2)$. But then $r_{\theta_1} e^{i\theta_1}$ and $r_{\theta_2} e^{i\theta_2}$ are distinct roots of the polynomial $t^m - \zeta(\theta_1)t - (1-\zeta(\theta_1))$, both with argument belonging to $(\frac{2\pi}{m}, \frac{2\pi}{m-1})$, which contradicts what we have obtained at the beginning of the proof of part(iii). As ζ is one-to-one and continuous, it is either strictly increasing or strictly decreasing on $(\frac{2\pi}{m}, \frac{2\pi}{m-1})$. In view of the intermediate value property of a real-valued continuous function, it suffices to show that $\lim_{\theta \to \frac{2\pi}{m}^+} \zeta(\theta) = 0$ and $\lim_{\theta \to \frac{2\pi}{m-1}^-} \zeta(\theta)$ equals 1 when $m \geq 4$ and equals $\frac{3}{4}$ when $m = 3$.

By the definition of ζ and the fact that $0 < r_\theta < 1$ it is readily seen that we have $\lim_{\theta \to \frac{2\pi}{m}^+} \zeta(\theta) = 0$. On the other hand, from the condition $g_\theta(r_\theta) = 0$ we also have $\zeta(\theta) = \frac{r_\theta^m \sin\,(m-1)\theta}{\sin\theta} + 1$, which implies that $\lim_{\theta \to \frac{2\pi}{m-1}^-} \zeta(\theta) = 1$, provided that $m \geq 4$. When $m = 3$, the polynomial equation given in (i) becomes $\frac{2^2}{3^3}t^3 = (t-1)^2$. Since $\frac{3}{4}$ is a root of the latter equation, we have $c_3 = \frac{3}{4}$. Note that the polynomial $g_\theta(t)$ tends to $-2t^3 - 3t^2 + 1$ as θ tends to π from the left. Also, $\frac{1}{2}$ is a root of the latter polynomial. So $\lim_{\theta \to \pi^-} r_\theta = \frac{1}{2}$, and we have

$$\lim_{\theta \to \pi^-} \zeta(\theta) = \lim_{\theta \to \pi} r_\theta^2 \frac{\sin 3\theta}{\sin\theta} = \frac{3}{4},$$

as desired. ■

In the course of the proof of Lemma 6.2 we have also established the following (see also [17, Lemma 3]):

Corollary 6.3. (i) *For any $\theta \in (\frac{2\pi}{m}, \frac{2\pi}{m-1}), r_\theta e^{i\theta}$, where r_θ is the unique positive real root of the polynomial $g_\theta(t)$ given in Lemma 6.2, is a root of the polynomial $t^m - ct - (1-c)$ where $c = r_\theta^{m-1}\frac{\sin m\theta}{\sin\theta}$. Moreover, the pair (r_θ, θ) satisfies the relations* (6.1) *and* (6.2)*, which appear in the proof of Lemma 6.2.*

(ii) *For every positive integer $m \geq 4$ (respectively, $m = 3$), every real number c in $(0,1)$ (respectively, in $(0, \frac{3}{4})$) can be expressed uniquely as $r_\theta^{m-1}\frac{\sin m\theta}{\sin\theta}$, where $\theta \in (\frac{2\pi}{m}, \frac{2\pi}{m-1})$.*

Theorem 6.4. (i) *For every positive integer $m \geq 3$,*

$$\max\{\gamma(K) : K \in \mathcal{P}(m,3)\} = 2m-1.$$

(ii) *For every $K \in \mathcal{P}(m,3)$, K is exp-maximal if and only if there exists a K-primitive matrix A such that the digraph $(\mathcal{E}, \mathcal{P}(A,K))$ is given by Figure 1.*

(iii) *Let $m \geq 3$ be a positive integer. For any $\theta \in (\frac{2\pi}{m}, \frac{2\pi}{m-1})$, let r_θ denote the unique positive real root of the polynomial $g_\theta(t)$ as defined in Lemma 6.2*(iii). *Let K_θ be the polyhedral cone in $\mathbb{R}^3$ generated by the vectors*

$$x_j(\theta) := \begin{bmatrix} r_\theta^{j-1}\cos(j-1)\theta \\ r_\theta^{j-1}\sin(j-1)\theta \\ 1 \end{bmatrix}, \ j = 1, \dots, m.$$

Also, let $A_\theta = r_\theta \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \oplus [1]$. Then $x_1(\theta), \dots, x_m(\theta)$ are the extreme vectors of K_θ, K_θ is an exp-maximal polyhedral cone and A_θ is an exp-maximal K_θ-primitive matrix.

Note that, for simplicity, we suppress the dependence of K_θ on m.

Proof. For every $K \in \mathcal{P}(m,3)$, by Corollary 4.9 (with $n = 3$) we have $\gamma(K) \leq 2m-1$, where the equality holds only if there exists a K-primitive matrix A such that the digraph $(\mathcal{E}, \mathcal{P})$ is given by Figure 1. In view of Lemma 6.1(ii), parts (i) and (ii) will follow if we can construct a polyhedral cone $K \in \mathcal{P}(m,3)$ for which there exists a K-primitive matrix A such that $(\mathcal{E}, \mathcal{P}(A,K))$ is given by Figure 1. To complete the proof, we are going to establish part (iii) and at the same time show that the digraph $(\mathcal{E}, \mathcal{P}(A_\theta, K_\theta))$ is given by Figure 1.

Consider any fixed $\theta \in (\frac{2\pi}{m}, \frac{2\pi}{m-1})$. Hereafter we denote r_θ simply by r. By Corollary 6.3(i) $re^{i\theta}$ is a root of the polynomial $t^m - ct - (1-c)$ where $c = r^{m-1}\frac{\sin m\theta}{\sin\theta}$. Consider the m points $y_1, \dots, y_m$ in $\mathbb{R}^2$ given by:

$$y_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } y_j = \begin{pmatrix} r^{j-1}\cos(j-1)\theta \\ r^{j-1}\sin(j-1)\theta \end{pmatrix} \text{ for } j = 2, \dots, m.$$

Take B to be the 2×2 matrix $r \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$. Note that $y_j = B^{j-1}y_1$ for $j = 2, \dots, m$. By Corollary 6.3(i) the equations (6.1) and (6.2), which appear in the

proof of Lemma 6.2, are satisfied. So we have

$$By_m = \begin{pmatrix} r^m \cos m\theta \\ r^m \sin m\theta \end{pmatrix} = c\begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix} + (1-c)\begin{pmatrix} 1 \\ 0 \end{pmatrix} = cy_2 + (1-c)y_1.$$

We contend that $y_1, \ldots, y_m$ are precisely all the extreme points of the convex polygon $C := \mathrm{conv}\{y_1, \ldots, y_m\}$, and moreover $\overline{y_m y_1}$ and $\overline{y_{i-1} y_i}$, for $i = 2, \ldots, m$, are precisely all its sides. Since $x_j(\theta) = \begin{pmatrix} y_j \\ 1 \end{pmatrix}$ for $j = 1, \ldots, m$, $A_\theta = B \oplus [1]$ and $BC \subseteq C$, once the contention is established, it is clear that $x_1(\theta), \ldots, x_m(\theta)$ are all the extreme vectors of K_θ, and A_θ is K_θ-nonnegative. Then we are done, as it can be readily checked that the digraph $(\mathcal{E}, \mathcal{P}(A_\theta, K_\theta))$ is given by Figure 1.

Clearly the extreme points of C are among $y_1, \ldots, y_m$. Since the Euclidean norm of y_1 is 1, whereas that of y_j, for $j = 2, \ldots, m$, is less than 1, y_1 is certainly an extreme point of C.

For $j = 2, \ldots, m$, proceeding inductively, we are going to show that y_j is an extreme point and $\overline{y_{j-1} y_j}$ forms a side of C. Once this is proved, it will follow that $\overline{y_m y_1}$ is also a side of C and we are done.

We begin with $j = 2$. If y_2 is not an extreme point of C, then y_2 lies in the relative interior of a line segment that joins two distinct points of C. But then y_3, which is By_2, also lies in the relative interior of a line segment joining two distinct points of C (as B is nonsingular and maps C into itself) and hence is not an extreme point of C. Continuing the argument, we conclude that $y_2, \ldots, y_m$ are all not extreme points of C; so y_1 is the only extreme point of C, which is impossible. This proves that y_2 must be an extreme point of C.

To proceed further, we need the following:

Assertion. Let M be a compact convex set in $\mathbb{R}^2$ which contains the origin as an interior point. Let u_1, u_2 be two distinct extreme points of M which are not negative multiples of each other. If the line segment $\overline{u_1 u_2}$ is not a face of M, then M has an extreme point of the form $\alpha_1 u_1 + \alpha_2 u_2$, where $\alpha_1, \alpha_2 > 0$.

Proof. Let L denote the straight line joining the points u_1, u_2. Since u_1, u_2 are not negative multiples of each other, the origin is on one side of the line. Move L parallel to itself but away from the origin until L becomes a supporting line L' for M. (Of course, the argument can be given more precisely in terms of a (continuous) linear functional.) Then $F := L' \cap M$ is a (boundary) face of M and hence is either an extreme point or a line segment. Note that the point $u_\lambda := \lambda(\frac{u_1+u_2}{2})$ belongs

to $\operatorname{int} M$, provided that $\lambda > 1$, sufficiently close to 1. Suppose that F has a point, say v, that does not belong to the interior of the cone $\operatorname{pos}\{u_1, u_2\}$. Since u_1, u_2 are extreme points of M and $0 \in M$, it is obvious that v cannot lie on the boundary of the said cone. So it lies outside the cone. But then the line segment $\overline{u_\lambda v}$ will meet one of the boundary rays of the said cone at a point of the form αu_j, where $\alpha > 1$ and $j = 1$ or 2, which contradicts the assumption that u_1, u_2 are extreme points of M. This shows that $F \subseteq \operatorname{int}(\operatorname{pos}\{u_1, u_2\})$. As each extreme point of F is also an extreme point of M, now it is clear that M has an extreme point of the desired form. ■

Now back to the proof of our theorem. In view of the definition of the y_i's, it is clear that none of the points $y_3, \ldots, y_m$ can be written as a positive linear combination of y_1 and y_2. (This can be seen, for instance, by considering the arguments of the complex numbers corresponding to these points.) So, by the above Assertion, the line segment $\overline{y_1 y_2}$ forms a side of the polygon C.

Note that for each $j, j = 3, \ldots, m-1$, by the preceding argument we have used for y_2, we can show that if y_j is not an extreme point of C, then none of the points $y_{j+1}, \ldots, y_m$ is an extreme point of C, and so we must have $C = \operatorname{conv}\{y_1, \ldots, y_{j-1}\}$. But C is not the convex hull of all the y_j's that lie on the upper closed half-plane, so it follows each of points $y_1, \ldots, y_k$ must be an extreme point of C, where we denote by k the first positive integer such that $k\theta > \pi$ (i.e., $k = \lfloor \frac{\pi}{\theta} \rfloor + 1$). Moreover, for $j = 3, \ldots, k$, once we have obtained that y_{j-1} and y_j are extreme points, like the case $j = 2$, using the above Assertion, we can infer that $\overline{y_{j-1} y_j}$ forms a side of the polygon C.

If our inductive argument should fail at one step, then there must exist some $s, k \leq s \leq m-1$, such that $y_1, \ldots, y_s$ are precisely all the extreme points of C and moreover each of the line segments $\overline{y_{j-1} y_j}$, for $j = 2, \ldots, s$, forms a side of C. Then the line segment $\overline{y_s y_1}$ is clearly the remaining side of C. Since y_s is in the lower open half-plane, a consideration of the arguments of the complex numbers corresponding to the points $y_1, \ldots, y_m$ shows that y_m must be a positive linear combination of y_1 and y_s. So we have either $y_m \in \operatorname{int} \operatorname{conv}\{y_1, y_s, 0\}$ or $y_m \in \operatorname{ri} \overline{y_1 y_s}$. In any case, y_s belongs to $\Phi(y_m)$, the face of C generated by y_m. Hence $y_{s+1} = By_s \in \Phi(By_m) = \overline{y_1 y_2}$, where the last equality holds as $By_m = (1-c)y_1 + cy_2 (0 < c < 1)$ and it has already been shown that $\overline{y_1 y_2}$ is a side (i.e., a face) of C. As $k+1 \leq s+1 \leq m$, we arrive at a contradiction.

So proceeding inductively, we show that $y_1, \ldots, y_m$ are all the extreme points of C and $\overline{y_{j-1} y_j}$ forms a side of C for $j = 2, \ldots, m$. The proof is complete. ■

In contrast to Theorem 6.4, we have the following:

Theorem 6.5. *For every positive integer $m \geq 5$ there exists a 3-dimensional polyhedral cone K with m extreme rays for which there is no K-primitive matrix A with the digraph $(\mathcal{E}, \mathcal{P}(A, K))$ given by Figure 1.*

Proof. Let K be the polyhedral cone in $\mathbb{R}^3$ with extreme vectors

$$y_j = (\cos\frac{2j\pi}{m}, \sin\frac{2j\pi}{m}, 1)^T, j = 1, \ldots, m.$$

We contend that there is no K-primitive matrix A for which $(\mathcal{E}, \mathcal{P})$ is given by Figure 1, where $x_1, x_2, \ldots, x_m$ is a rearrangement of $y_1, \ldots, y_m$.

We assume to the contrary that there is one such A. By Lemma 6.1, for $i = 1, \ldots, m, x_i$ and x_{i+1} are neighborly extreme vectors of K (where x_{m+1} is taken to be x_1). On the other hand, for each j, the extreme vectors neighborly to y_j are y_{j+1} and y_{j-1}. [We adopt the convention that for each integer j, y_j equals y_k where k is the unique integer that satisfies $1 \leq k \leq m, k \equiv j \pmod{m}$.] Suppose $x_1 = y_{j_1}$, where $1 \leq j_1 \leq m$. Since x_2 is neighborly to x_1, x_2 must be either y_{j_1+1} or y_{j_1-1}. Consider first the case when $x_2 = y_{j_1+1}$. Since x_3 is neighborly to x_2, it is equal to either y_{j_1+2} or y_{j_1}. But we already have $x_1 = y_{j_1}$, so x_3 must be y_{j_1+2}. Continuing the argument, we can show that $x_j = y_{j_1+j-1}$ for $j = 1, \ldots, m$. Then we take $\hat{A}$ to be $\begin{bmatrix} \cos\frac{2\pi}{m} & -\sin\frac{2\pi}{m} \\ \sin\frac{2\pi}{m} & \cos\frac{2\pi}{m} \end{bmatrix} \oplus [1]$. If $x_2 = y_{j_1-1}$, we can show in a similar manner that $x_j = y_{j_1-j+1}$ for $j = 1, \ldots, m$. In this case, we take $\hat{A}$ to be $\begin{bmatrix} \cos\frac{2\pi}{m} & \sin\frac{2\pi}{m} \\ -\sin\frac{2\pi}{m} & \cos\frac{2\pi}{m} \end{bmatrix} \oplus [1]$. As we are going to show, in either case, $\hat{A}$ and A are positive multiples of each other.

It is known that if C_1, C_2 are proper cones in $\mathbb{R}^{n_1}$ and $\mathbb{R}^{n_2}$ respectively then the set $\pi(C_1, C_2)$ which consists of all $n_2 \times n_1$ matrices B such that $BC_1 \subseteq C_2$ is a proper cone in the space of $n_2 \times n_1$ real matrices.

Now let K_1 denote the 3-dimensional polyhedral cone $\text{pos}\{x_1, x_2, x_3, x_4\}$. It is clear that $A \in \pi(K_1, K)$. As the digraph $(\mathcal{E}, \mathcal{P})$ is given by Figure 1 and $m \geq 5, Ax_i$ is a positive multiple of x_{i+1} for $i = 1, 2, 3, 4$. So A maps every extreme vector of K_1 to an extreme vector of K. But K_1 is indecomposable and A is nonsingular, so by a variant of a sufficient condition for a cone-preserving map to be extreme due to Loewy and Schneider [19, Theorem 3.3] (see [21, Theorem 1] or [32, Theorem 5.2]), it follows that A is an extreme matrix of the proper cone $\pi(K_1, K)$. Our definition of $\hat{A}$ guarantees that $\hat{A}x_i = x_{i+1}$ for $i = 1, \ldots, 4$; hence $\hat{A}x_i$ is a positive multiple

of Ax_i for $i = 1, \ldots, 4$. But $\{x_1, \ldots, x_4\}$ is the set of extreme vectors of K_1, so $\hat{A}$ belongs to the extreme ray of $\pi(K_1, K)$ generated by A. Hence A and $\hat{A}$ are positive multiples of each other. But then the digraph $(\mathcal{E}, \mathcal{P}(\hat{A}, K))$ is not given by Figure 1 (and also $\hat{A}$ is not K-primitive). So we arrive at a contradiction. ■

Let K_1, K_2 be linearly isomorphic proper cones. If D is a digraph that can be realized as $(\mathcal{E}, \mathcal{P}(A_1, K_1))$ for some K_1-nonnegative matrix A_1, then clearly (up to graph isomorpism) D can also be realized as $(\mathcal{E}, \mathcal{P}(A_2, K_2))$ for some K_2-nonnegative matrix A_2. On the other hand, if K_1, K_2 are assumed to be combinatorially equivalent only, then the same cannot be said.

Remark 6.6. Let K_1, K_2 be combinatorially equivalent proper cones. Then:

(i) If G is a digraph such that $G = (\mathcal{E}(K_1), \mathcal{P}(A_1, K_1))$ for some K_1-primitive matrix A_1, then there need not exist a K_2-primitive matrix A_2 such that $(\mathcal{E}(K_2), \mathcal{P}(A_2, K_2))$ is isomorphic with G.

(ii) The values of $\gamma(K_1), \gamma(K_2)$ need not be the same.

Since any two 3-dimensional polyhedral cones with the same number of extreme rays are combinatorially equivalent, the preceding remark follows from Theorem 6.4 and Theorem 6.5.

7. The higher-dimensional case

In this section we are going to establish the following main result of our paper.

Theorem 7.1. *For any pair of positive integers $m, n, 3 \leq n \leq m$,* $\max\{\gamma(K) : K \in \mathcal{P}(m, n)\}$ *equals $(n-1)(m-1)+1$ when m is even or m and n are both odd, and is at least $(n-1)(m-1)$ and at most $(n-1)(m-1)+1$ when m is odd and n is even.*

First, we explain how the proof starts. By Corollary 4.9 the inequality $\gamma(K) \leq (n-1)(m-1)+1$ holds for any $K \in \mathcal{P}(m, n)$. By the same corollary, when the inequality becomes equality, there exists a K-primitive matrix A such that the digraph $(\mathcal{E}, \mathcal{P}(A, K))$ is given by Figure 1. Then a straightforward computation shows that $A^{(n-1)(m-1)-1}x_1$ is a positive linear combination of the $n-1$ extreme vectors $x_{m-n+1}, x_{m-n+2}, \ldots, x_{m-1}$, whereas $A^{(n-1)(m-1)}x_1$ is a positive linear combination

of the $n-1$ extreme vectors $x_{m-n+2}, x_{m-n+3}, \ldots, x_{m-1}, x_m$. On the other hand, by Lemma 4.2(ii) we also have $\gamma(A, x_1) = \gamma(A) \leq (n-1)(m-1)+1$. Note that the latter inequality implies that $A^{(n-1)(m-1)+1}x_1 \in \operatorname{int} K$. So, to establish Theorem 7.1, it suffices to show that when m is even or when m and n are both odd (respectively, when m is odd and n is even), it is possible to construct a polyhedral cone $K \in \mathcal{P}(m,n)$ and a K-primitive matrix A such that the digraph $(\mathcal{E}, \mathcal{P}(A,K))$ is given by Figure 1 and $x_{m-n+2}+x_{m-n+3}+\cdots+x_m \in \partial K$ (respectively, $x_{m-n+1}+x_{m-n+2}+\cdots+x_{m-1} \in \partial K$).

To begin with, we show that for every pair $m, n, 3 \leq n \leq m$, the digraph given by Figure 1 can always be realized as $(\mathcal{E}, \mathcal{P}(A,K))$ for some $K \in \mathcal{P}(m,n)$ and a K-primitive matrix A.

Lemma 7.2. *For every pair of positive integers $m, n, 3 \leq n \leq m$, there is a polyhedral cone $K \in \mathcal{P}(m,n)$ for which there exists a K-primitive matrix A such that the digraph $(\mathcal{E}, \mathcal{P}(A,K))$ is given by Figure* 1.

Proof. First, we treat the case when m, n are both odd. Write $m = 2k+1$ and $n = 2p+1$. Then $1 \leq p \leq k$. Choose any $c \in (0, c_m)$, where $c_m \in (0,1)$ is the positive real number defined in Lemma 6.2, and let $h(t)$ denote the polynomial $t^m - ct - (1-c)$. By part (iii) of the said lemma $h(t)$ has a complex root of the form $r_1 e^{i\theta_1}$, where $\theta_1 \in (\frac{2\pi}{m}, \frac{2\pi}{m-1})$ and $0 < r_1 < 1$. In addition, $h(t)$ has no real roots other than 1. So $h(t)$ has k pairs of non-real conjugate complex roots, say, $r_j e^{\pm i\theta_j}, j = 1, \ldots, k$ (where $r_1 e^{i\theta_1}$ is the one already mentioned). Let K be the polyhedral cone in $\mathbb{R}^n$ given by $K = \operatorname{pos}\{x_1, \ldots, x_m\}$, where for $j = 1, \ldots, m$,

$$x_j = \begin{bmatrix} r_1^{j-1}\cos(j-1)\theta_1 \\ r_1^{j-1}\sin(j-1)\theta_1 \\ \vdots \\ r_p^{j-1}\cos(j-1)\theta_p \\ r_p^{j-1}\sin(j-1)\theta_p \\ 1 \end{bmatrix}.$$

It is clear that K is a pointed cone. A sufficient condition for K to be a full cone is that the $n \times n$ matrix whose jth column is x_j, for $j = 1, \ldots, n$, is nonsingular. Upon pre-multiplying the latter matrix by the $n \times n$ matrix

$$\begin{bmatrix} 1 & i \\ 1 & -i \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} 1 & i \\ 1 & -i \end{bmatrix} \oplus [1],$$

we obtain the Vandermonde matrix

$$\begin{bmatrix} 1 & z_1 & \cdots & z_1^{n-1} \\ 1 & \bar{z_1} & \cdots & \bar{z_1}^{n-1} \\ \vdots & \vdots & \vdots & \vdots \\ 1 & z_p & \cdots & z_p^{n-1} \\ 1 & \bar{z_p} & \cdots & \bar{z_p}^{n-1} \\ 1 & 1 & \cdots & 1 \end{bmatrix},$$

where $z_j = r_j e^{i\theta_j}$ for $j = 1, \ldots, p$, which is nonsingular, as the roots of the polynomial $h(t)$ are simple (see Lemma 6.2). So K is a full cone.

Now take A to be the $n \times n$ matrix

$$r_1 \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{bmatrix} \oplus \cdots \oplus r_p \begin{bmatrix} \cos\theta_p & -\sin\theta_p \\ \sin\theta_p & \cos\theta_p \end{bmatrix} \oplus [1].$$

As can be readily checked, $Ax_j = x_{j+1}$ for $j = 1, \ldots, m-1$. Also, we have

$$Ax_m = \begin{bmatrix} r_1^m \cos m\theta_1 \\ r_1^m \sin m\theta_1 \\ \vdots \\ r_p^m \cos m\theta_p \\ r_p^m \sin m\theta_p \\ 1 \end{bmatrix}.$$

But the assumption that $r_j e^{i\theta_j}$ are roots of $h(t)$ for $j = 1, \ldots, p$ implies that relations (6.1) and (6.2), which appear in the proof of Lemma 6.2, are satisfied for the pair $(r, \theta) = (r_j, \theta_j)$ for every such j. So we have $Ax_m = (1-c)x_1 + cx_2$. Hence A is K-nonnegative. It remains to show that $x_1, \ldots, x_m$ are precisely the pairwise distinct extreme vectors of K (the polyhedral cone generated by them) and the face $\Phi(x_1 + x_2)$ contains (up to multiples) only x_1, x_2 as its extreme vectors. Once this is done, it will follow that the digraph $(\mathcal{E}, \mathcal{P})$ is given by Figure 1.

For $j = 1, \ldots, m$, denote by u_j the subvector of x_j formed by its 1st, 2nd and last components. Since $\theta_1 \in (\frac{2\pi}{m}, \frac{2\pi}{m-1})$ and $r_1 = r_{\theta_1}$, by Theorem 6.4(iii) $u_1, \ldots, u_m$ are precisely the pairwise distinct extreme vectors of the polyhedral cone $\text{pos}\{u_1, \ldots, u_m\}$. So it is clear that each x_j cannot be written as a nonnegative linear combination of the remaining x_l's or, in other words, $x_1, \ldots, x_m$ are precisely the extreme vectors of K. Also, the proof of Theorem 6.4(iii) guarantees that u_1, u_2 are

neighborly extreme vectors of the 3-dimensional polyhedral cone $\mathrm{pos}\{u_1, \ldots, u_m\}$, which means that there is no representation of $u_1 + u_2$ as a positive linear combination of $u_1, \ldots, u_m$, in which at least one of the vectors $u_3, \ldots, u_m$ is involved. As a consequence, there is also no representation of $x_1 + x_2$ as a positive linear combination of $x_1, \ldots, x_m$, in which at least one of the vectors $x_3, \ldots, x_m$ is involved. Hence, the face of K generated by $x_1 + x_2$ is 2-dimensional, as desired.

Now we consider the problem of constructing the desired pair (K, A) for even m. Choose any $c \in (0, 1)$. By Lemma 6.2 the polynomial $h(t) = t^m - ct - (1-c)$ has a complex root $r_1 e^{i\theta_1}$ with $\theta_1 \in (\frac{2\pi}{m}, \frac{2\pi}{m-1})$ and $0 < r_1 < 1$. Furthermore, $h(t)$ has two distinct real roots, namely, 1 and, say a. We write m as $2k+2$ and let the non-real complex roots of $h(t)$ be $r_j e^{\pm i\theta_j}$ for $j = 1, \ldots, k$ (where $r_1 e^{i\theta_1}$ is the root just mentioned). Now write n as $2p+2$ or $2p+1$ (with $1 \leq p \leq k$), depending on whether n is even or odd. Let K be the polyhedral cone in $\mathbb{R}^n$ given by:

$$K = \mathrm{pos}\{x_1, \ldots, x_m\},$$

where for $j = 1, \ldots, m$,

$$x_j = \begin{bmatrix} r_1^{j-1}\cos(j-1)\theta_1 \\ r_1^{j-1}\sin(j-1)\theta_1 \\ \vdots \\ r_p^{j-1}\cos(j-1)\theta_p \\ r_p^{j-1}\sin(j-1)\theta_p \\ a^{j-1} \\ 1 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} r_1^{j-1}\cos(j-1)\theta_1 \\ r_1^{j-1}\sin(j-1)\theta_1 \\ \vdots \\ r_p^{j-1}\cos(j-1)\theta_p \\ r_p^{j-1}\sin(j-1)\theta_p \\ 1 \end{bmatrix},$$

depending on whether n is even or odd.

Now take A to be the $n \times n$ matrix

$$r_1 \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{bmatrix} \oplus \cdots \oplus r_p \begin{bmatrix} \cos\theta_p & -\sin\theta_p \\ \sin\theta_p & \cos\theta_p \end{bmatrix} \oplus [a] \oplus [1]$$

or

$$r_1 \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{bmatrix} \oplus \cdots \oplus r_p \begin{bmatrix} \cos\theta_p & -\sin\theta_p \\ \sin\theta_p & \cos\theta_p \end{bmatrix} \oplus [1],$$

again depending on whether n is even or odd. Using the same argument as before, we can show that K is a proper cone, $x_1, \ldots, x_m$ are its extreme vectors, A is K-primitive and $(\mathcal{E}, \mathcal{P})$ is given by Figure 1.

When m is odd and n is even, take any $c \in (c_m, 1)$. According to Lemma 6.2, the polynomial $h(t) = t^m - ct - (1-c)$ has simple roots, three of which are real, namely, 1 and say a_1, a_2, with $a_1 < a_2$, where $a_1, a_2 \in (-1, 0)$. Let the remaining $2k$ (where $m - 3 = 2k$) non-real roots be $r_j e^{\pm i\theta_j}, j = 1, \ldots, k$. (Note that in this case we have $m \geq 5$.) By the same lemma, we may assume that $\theta_1 \in (\frac{2\pi}{m}, \frac{2\pi}{m-1})$. Write n as $2p + 2$. Then $1 \leq p \leq k$. Let K be the polyhedral cone in $\mathbb{R}^n$ given by:

$$K = \mathrm{pos}\{x_1, \ldots, x_m\},$$

where for $j = 1, \ldots, m$,

$$x_j = \begin{bmatrix} r_1^{j-1} \cos(j-1)\theta_1 \\ r_1^{j-1} \sin(j-1)\theta_1 \\ \vdots \\ r_p^{j-1} \cos(j-1)\theta_p \\ r_p^{j-1} \sin(j-1)\theta_p \\ a_1^{j-1} \\ 1 \end{bmatrix}.$$

Now let A be the matrix

$$r_1 \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{bmatrix} \oplus \cdots \oplus r_p \begin{bmatrix} \cos\theta_p & -\sin\theta_p \\ \sin\theta_p & \cos\theta_p \end{bmatrix} \oplus [a_1] \oplus [1].$$

The subsequent arguments are similar to those for the previous cases. We omit the details. ■

Our next step is to show the following crucial lemma.

Lemma 7.3. *Let m, n be positive integers such that $3 \leq n \leq m$, where n is odd or n, m are both even. Let K_0 be the cone in $\mathbb{R}^n$ with extreme vectors $y_1, \ldots, y_m$ given by* :

$$y_j = \begin{bmatrix} \cos(j-1)\frac{2\pi}{m} \\ \sin(j-1)\frac{2\pi}{m} \\ \cos(j-1)\frac{4\pi}{m} \\ \sin(j-1)\frac{4\pi}{m} \\ \vdots \\ \cos(j-1)\frac{(n-1)\pi}{m} \\ \sin(j-1)\frac{(n-1)\pi}{m} \\ 1 \end{bmatrix}, 1 \leq j \leq m,$$

when n is odd ; and by

$$y_j = \begin{bmatrix} \cos(j-1)\frac{2\pi}{m} \\ \sin(j-1)\frac{2\pi}{m} \\ \cos(j-1)\frac{4\pi}{m} \\ \sin(j-1)\frac{4\pi}{m} \\ \vdots \\ \cos(j-1)\frac{(n-2)\pi}{m} \\ \sin(j-1)\frac{(n-2)\pi}{m} \\ (-1)^{j-1} \\ 1 \end{bmatrix}, 1 \le j \le m,$$

when n, m are both even. Then $\sum_{j=1}^{n-1} y_j$ lies in ∂K_0 and generates a simplicial face of K_0.

Before we consider the proof of Lemma 7.3, we show how it can be used to finish the proof of Theorem 7.1.

We first treat the case when m is even or m, n are both odd. Recall the proof of Lemma 7.2. In constructing the polyhedral cone K and the K-primitive matrix A, certain real numbers r_j's and θ_j's are involved. For the argument to work there, $r_j e^{\pm i\theta_j}$ ($j = 1, \dots, p$, where $p = \frac{n-1}{2}$ or $\frac{n-2}{2}$, depending on whether n is odd or even) can be any p pairs of non-real conjugate complex roots of the polynomial $h(t)$, but θ_1 has to be chosen from $(\frac{2\pi}{m}, \frac{2\pi}{m-1})$. Now in the proof of Theorem 7.1 in order that the argument works we need to be more careful about the choices of the r_j's and θ_j's. Since $h(t)$ tends to the polynomial $t^m - 1$ as c tends to 0 (from the right), when c is sufficiently close to 0, for each $j = 1, \dots, \frac{m}{2} - 1$, there is precisely one root of $h(t)$ within an ε-neighborhood of $e^{\pm \frac{2\pi j}{m} i}$, where ε is a sufficiently small positive number such that the ε-neighborhoods of $e^{\pm \frac{2\pi j}{m} i}$ for $j = 1, \dots, \frac{m}{2}$ are pairwise mutually disjoint. We label that particular root as $r_j e^{i\theta_j}$. Then it is readily seen that the extreme vectors $x_1, \dots, x_m$ of K can be made as close as we please to, respectively, the extreme vectors $y_1, \dots, y_m$ of K_0. By Lemma 7.3 the vector $y_1 + y_2 + \cdots + y_{n-1}$ lies in ∂K_0 and generates a simplicial face of K_0. The same is also true for the vector $y_{m-n+2} + y_{m-n+3} + \cdots + y_m$ of K_0, as it is clear that there is an automorphism of K_0 that takes y_j to $y_{j+m-n+1}$ for $j = 1, \dots, m$ (where for $r > m, y_r$ is taken to be y_s with $s \equiv r (\operatorname{mod} m), 1 \le s \le m$). Hence $\operatorname{span}\{y_{m-n+2}, y_{m-n+3}, \dots, y_m\}$ is a supporting hypersubspace for K_0 and the remaining extreme vectors $y_1, \dots, y_{m-n+1}$ all lie in the same open half-space determined by this hypersubspace. By continuity,

it follows that when c is sufficiently close to 0, $\mathrm{span}\{x_{m-n+2}, x_{m-n+3}, \ldots, x_m\}$ is a supporting hypersubspace for K and the remaining extreme vectors of K all lie on the same open half-space determined by this hypersubspace. The latter implies that $x_{m-n+2} + x_{m-n+3} + \cdots + x_m \in \partial K$, which is what we want.

When m is odd and n is even, we still use the construction for K and A as given in the proof of Lemma 7.2. However, instead of K_0 we make use of the polyhedral cone K_1 of $\mathbb{R}^n$ with extreme vectors $y_1, \ldots, y_{m-1}$ given by:

$$y_j = \begin{bmatrix} \cos(j-1)\frac{2\pi}{m-1} \\ \sin(j-1)\frac{2\pi}{m-1} \\ \cos(j-1)\frac{4\pi}{m-1} \\ \sin(j-1)\frac{4\pi}{m-1} \\ \vdots \\ \cos(j-1)\frac{(n-2)\pi}{m-1} \\ \sin(j-1)\frac{(n-2)\pi}{m-1} \\ (-1)^{j-1} \\ 1 \end{bmatrix}, 1 \le j \le m-1.$$

In defining the x_j's and the matrix A, we choose the θ_j's ($j = 1, \ldots, \frac{n-2}{2}$) in such a way that $r_j e^{\pm i\theta_j}$ is the root of $h(t)$ closest to $e^{\pm \frac{2\pi j}{m-1} i}$, noting that $r_j e^{i\theta_j}, r_j e^{-i\theta_j}, j = 1, \ldots, \frac{m-3}{2}, a_1, a_2$ and 1, which are the roots of the polynomial $h(t)$, approach to, respectively, the roots $e^{\frac{2\pi j}{m-1} i}, e^{-\frac{2\pi j}{m-1} i}, j = 1, \ldots, \frac{m-3}{2}, -1, 0$ and 1 of the polynomial $t^m - t$, as c approach to 1 (from the left). Then x_j tends to y_j for $j = 1, \ldots, m-1$ and also x_m tends to y_1 as c tends to 1. Since $m-1$ is even, by what we have just done before (for the case when m is even), the vector $y_{m-n+1} + y_{m-n+2} + \ldots + y_{m-1}$ generates an $(n-1)$-dimensional simplicial face of K_1. So $\mathrm{span}\{y_{m-n+1}, y_{m-n+2}, \ldots, y_{m-1}\}$ is a supporting hypersubspace for K_1 and the remaining extreme vectors $y_1, \ldots, y_{m-n}$ all lie in the same open half-space determined by this hypersubspace. By continuity, when c is sufficiently close to 1, $\mathrm{span}\{x_{m-n+1}, x_{m-n+2}, \ldots, x_{m-1}\}$ is a supporting hypersubspace for K and the remaining extreme vectors of K all lie in the same open half-space determined by this hypersubspace (noting that x_m also lies in this open half-space as x_m, x_1 both tend to y_1). Hence, $x_{m-n+1} + x_{m-n+2} + \ldots + x_{m-1} \in \partial K$, as desired.

Now we come to the proof of Lemma 7.3. The proof is fairly long and takes several steps.

The lemma clearly holds for the case $m = n$. Hereafter we assume that $m > n$.

We want to show that $\sum_{j=1}^{n-1} y_j$ lies in ∂K_0 and generates a simplicial face. For the purpose, it suffices to find a vector v such that $\langle v, y_j \rangle$ equals 0 for $j = 1, \ldots, n-1$ and is positive for $j = n, \ldots, m$. Note that the vectors $y_1, y_2, \ldots, y_{n-1}$ are linearly independent, because by the argument given in the proof of Lemma 7.2 one can show that the $n \times n$ matrix

$$P := \begin{bmatrix} y_1 & y_2 & \cdots & y_n \end{bmatrix}$$

is nonsingular. So the desired vector v is, up to a positive scalar multiple, uniquely determined. Let $\tilde{v} = (C_{1n}, \ldots, C_{nn})^T$, where C_{ln} denotes the (l, n)-cofactor of P. By elementary properties of the determinant function, for each p, we have $\langle \tilde{v}, y_p \rangle = \det Q_p$, where Q_p denotes the $n \times n$ matrix obtained from P by replacing its nth column by y_p. Note that $\langle v, y_p \rangle = 0$ for $p = 1, \ldots, n-1$. It remains to show that for $p = n, n+1, \ldots, m, \det Q_p$ are all nonzero and have the same sign — if $\det Q_p$'s are all positive, take $v = \tilde{v}$; if $\det Q_p$'s are all negative, take $v = -\tilde{v}$.

We find it more convenient to work in the complex domain. Denote $e^{2\pi i/m}$ by ω. Pre-multiplying the matrix Q_p by

$$\underbrace{\begin{bmatrix} 1 & i \\ 1 & -i \end{bmatrix} \oplus \begin{bmatrix} 1 & i \\ 1 & -i \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} 1 & i \\ 1 & -i \end{bmatrix}}_{\frac{n-2}{2} \text{ times}} \oplus I_2$$

when n is even, or by

$$\underbrace{\begin{bmatrix} 1 & i \\ 1 & -i \end{bmatrix} \oplus \begin{bmatrix} 1 & i \\ 1 & -i \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} 1 & i \\ 1 & -i \end{bmatrix}}_{\frac{n-1}{2} \text{ times}} \oplus (1)$$

when n is odd, we obtain the matrix

$$\begin{bmatrix} 1 & \omega & \omega^2 & \cdots & \omega^{n-2} & \omega^{p-1} \\ 1 & \bar{\omega} & \bar{\omega}^2 & \cdots & \bar{\omega}^{n-2} & \bar{\omega}^{p-1} \\ 1 & \omega^2 & \omega^4 & \cdots & \omega^{2(n-2)} & \omega^{2(p-1)} \\ 1 & \bar{\omega}^2 & \bar{\omega}^4 & \cdots & \bar{\omega}^{2(n-2)} & \bar{\omega}^{2(p-1)} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & \omega^{\frac{n-2}{2}} & \omega^{n-2} & \cdots & \omega^{\frac{(n-2)^2}{2}} & \omega^{\frac{(n-2)(p-1)}{2}} \\ 1 & \bar{\omega}^{\frac{n-2}{2}} & \bar{\omega}^{n-2} & \cdots & \bar{\omega}^{\frac{(n-2)^2}{2}} & \bar{\omega}^{\frac{(n-2)(p-1)}{2}} \\ 1 & -1 & 1 & \cdots & (-1)^{n-2} & (-1)^{p-1} \\ 1 & 1 & 1 & \cdots & 1 & 1 \end{bmatrix}$$

when m, n are both even, or the matrix

$$\begin{bmatrix} 1 & \omega & \omega^2 & \cdots & \omega^{n-2} & \omega^{p-1} \\ 1 & \bar{\omega} & \bar{\omega}^2 & \cdots & \bar{\omega}^{n-2} & \bar{\omega}^{p-1} \\ 1 & \omega^2 & \omega^4 & \cdots & \omega^{2(n-2)} & \omega^{2(p-1)} \\ 1 & \bar{\omega}^2 & \bar{\omega}^4 & \cdots & \bar{\omega}^{2(n-2)} & \bar{\omega}^{2(p-1)} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & \omega^{\frac{n-1}{2}} & \omega^{n-1} & \cdots & \omega^{\frac{(n-1)(n-2)}{2}} & \omega^{\frac{(n-1)(p-1)}{2}} \\ 1 & \bar{\omega}^{\frac{n-1}{2}} & \bar{\omega}^{n-1} & \cdots & \bar{\omega}^{\frac{(n-1)(n-2)}{2}} & \bar{\omega}^{\frac{(n-1)(p-1)}{2}} \\ 1 & 1 & 1 & \cdots & 1 & 1 \end{bmatrix}$$

when n is odd. Denote the determinant of the matrix by e_p.

Note that for each $p = n, n+1, \ldots, m, e_p$ is equal to $\det Q_p$ times a nonzero constant, which depends on n but not on p. More specifically, the said nonzero constant is $\pm 2^{\frac{n-2}{2}}$ if $n \equiv 2 \pmod 4$, $\pm 2^{\frac{n-2}{2}} i$ if $n \equiv 0 \pmod 4$, $\pm 2^{\frac{n-1}{2}}$ if $n \equiv 1 \pmod 4$, and $\pm 2^{\frac{n-1}{2}} i$ if $n \equiv 3 \pmod 4$. We want to show all e_ps are nonzero real numbers with the same sign if $n \equiv 2 \pmod 4$ or $n \equiv 1 \pmod 4$, and if $n \equiv 0 \pmod 4$ or $n \equiv 3 \pmod 4$ then all ie_ps are also nonzero real numbers with the same sign.

We now consider a kind of generalized Vandermonde determinant on the indeterminates $t_1, \ldots, t_n$. Let $n \geq 3$. For every integer $p \geq n-1$, let $f_p(t_1, \ldots, t_n)$ denote

the polynomial function

$$f_p(t_1, \ldots, t_n) = \det \begin{bmatrix} 1 & t_1 & t_1^2 & \cdots & t_1^{n-2} & t_1^p \\ 1 & t_2 & t_2^2 & \cdots & t_2^{n-2} & t_2^p \\ 1 & t_3 & t_3^2 & \cdots & t_3^{n-2} & t_3^p \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & t_n & t_n^2 & \cdots & t_n^{n-2} & t_n^p \end{bmatrix}.$$

By the kth *complete* (*homogeneous*) *symmetric polynomial* in $t_1, \ldots, t_n$, denoted by $h_k(t_1, t_2, \ldots, t_n)$, we mean the polynomial which is the sum of all monomials in $t_1, \ldots, t_n$ of degree k, where each monomial appears exactly once. For instance,

$$h_2(t_1, t_2, t_3) = t_1t_2 + t_1t_3 + t_2t_3 + t_1^2 + t_2^2 + t_3^2.$$

By definition, $h_0(t_1, \ldots, t_n) \equiv 1$. It is convenient to define $h_r(t_1, \ldots, t_n)$ to be 0 for $r < 0$.

Claim 1. For every integer $p \geq n - 1$, we have

$$f_p(t_1, t_2, \ldots, t_n) = h_{p+1-n}(t_1, t_2, \ldots, t_n) \prod_{1 \leq i < j \leq n} (t_j - t_i). \tag{7.1}$$

The above claim can be deduced from a formula that expresses a Schur function $s_\lambda(t_1, \ldots, t_n)$ as a determinant involving complete symmetric polynomials $h_r(t_1, \ldots, t_n)$, known as the *Jacobi-Trudi determinant* (see Macdonald [22, p.25] or Sagan [26, p.154-159]). Recall that if $\lambda = (\lambda_1, \lambda_2, \ldots)$ is a *partition* of length $\leq n$ (i.e. a finite or infinite sequence of nonnegative integers in non-increasing order such that the number of nonzero λ_i is at most n) then the quotient

$$\frac{\det[t_i^{j-1+\lambda_{n-j+1}}]_{1 \leq i,j \leq n}}{\prod_{1 \leq i < j \leq n}(t_j - t_i)}$$

is a symmetric polynomial in $t_1, \ldots, t_n$. It is called the *Schur function associated with* λ and is denoted by $s_\lambda(t_1, \ldots, t_n)$. For $p \geq n-1$, the polynomial $f_p(t_1, \ldots, t_n)$ introduced above satisfies

$$\frac{f_p(t_1, \ldots, t_n)}{\prod_{1 \leq i < j \leq n}(t_j - t_i)} = s_\lambda(t_1, \ldots, t_n),$$

where λ is the partition $(p-(n-1),0,\ldots)$. According to the said determinantal formula, for any partition λ of length $\leq n$, we have

$$s_\lambda(t_1,\ldots,t_n)=\det[h_{\lambda_i-i+j}(t_1,\ldots,t_n)]_{1\leq i,j\leq n}.$$

A little calculation shows that, when $\lambda=(p-(n-1),0,\ldots)$, $[h_{\lambda_i-i+j}]_{1\leq i,j\leq n}$ is an upper triangular matrix whose $(1,1)$-entry is $h_{p-(n-1)}$ and all of whose other diagonal entries are equal to 1. It follows that we have $s_\lambda(t_1,\ldots,t_n)=h_{p+1-n}(t_1,\ldots,t_n)$, as desired. ■

For any set S of n complex numbers, we denote by $h_j(S)$ the value of $h_j(t_1,\ldots,t_n)$ evaluated at an n-tuple formed by all of the members of S, taken in any order. Similarly, we use $\sigma_j(S)$ to denote the *jth elementary symmetric function* on S.

Claim 2. Let S be a nonempty proper subset of $\mathbb{Z}_m$, the set of all mth roots of unity. Denote by S^c the complement of S in $\mathbb{Z}_m$. Then

$$h_j(S)=(-1)^j\sigma_j(S^c)$$

for $j=1,\ldots,m-n$, where n is the cardinality of S.

Proof. For any set T of n complex numbers, as is known (see, for instance, [22]), the generating function for the elementary symmetric functions on T is given by:

$$E(t;T)=\sum_{r=0}^{n}\sigma_r(T)t^r=\prod_{x\in T}(1+xt),$$

(where $\sigma_0(T)$ is taken to be 1). Also, the generating function for the complete symmetric polynomials on T is given by:

$$H(t;T)=\sum_{r=0}^{\infty}h_r(T)t^r=\prod_{x\in T}\frac{1}{1-xt}.$$

In view of the relation $\prod_{x\in\mathbb{Z}_m}(1-xt)=1-t^m$, for the given set S, we have

$$\sum_{r=0}^{m-n}(-1)^r\sigma_r(S^c)t^r=E(-t,S^c)=\prod_{x\in S^c}(1-xt)=\frac{1-t^m}{\prod_{x\in S}(1-xt)}=(1-t^m)\Sigma_{r=0}^{\infty}h_r(S)t^r.$$

By comparing the coefficients, our claim follows. ■

In view of Claim 1 and the discussion preceding the claim, it remains to show the following:

Claim 3. For $r = 1, \ldots, m-n$, $h_r(S) > 0$, where S is the subset of the set of mth roots of unity given by:

$$S = \begin{cases} \{\omega, \bar{\omega}, \ldots, \omega^{\frac{n-2}{2}}, \bar{\omega}^{\frac{n-2}{2}}, -1, 1\} \text{ when } m, n \text{ are both even} \\ \{\omega, \bar{\omega}, \ldots, \omega^{\frac{n-1}{2}}, \bar{\omega}^{\frac{n-1}{2}}, 1\} \text{ when } n \text{ is odd.} \end{cases}$$

Now, by Claim 2 we have

$$\prod_{x \in S^c} (t - x) = t^{m-n} + \sum_{j=1}^{m-n} (-1)^j \sigma_j(S^c) t^{m-n-j} = t^{m-n} + \sum_{j=1}^{m-n} h_j(S) t^{m-n-j}.$$

To establish Claim 3, it suffices to show that the coefficients of the polynomial $\prod_{x \in S^c}(t - x)$ are all positive. We complete our argument by applying the following interesting nontrivial result due to Barnard et al. [3, Theorem 1]. (Or see [4, Theorem 2.4.5] or [11, Theorem 4]).

Theorem D. *Let $p(t)$ be a polynomial of degree n, $p(0) = 1$, with nonnegative coefficients and zeros $a_1, \ldots, a_n$. For $\tau \geq 0$ write*

$$p_\tau(t) = \prod_{\substack{1 \leq j \leq n \\ |\arg(a_j)| > \tau}} (1 - t/a_j),$$

where $\arg(z)$ *is defined so that* $\arg(z) \in [-\pi, \pi)$. *If $p_\tau(t) \neq p(t)$, then the coefficients of $p_\tau(t)$ are all positive.*

Let us consider the case when m, n are both even first. In this case we have $S^c = \{\omega^{\frac{n}{2}}, \bar{\omega}^{\frac{n}{2}}, \ldots, \omega^{\frac{m}{2}-1}, \bar{\omega}^{\frac{m}{2}-1}\}$.Take $p(t)$ to be the following polynomial, which has nonnegative coefficients and constant term 1:

$$\prod_{\substack{1 \leq j \leq m-1 \\ j \neq \frac{m}{2}}} (t - \omega^j) = \frac{t^m - 1}{(t-1)(t+1)} = t^{m-2} + t^{m-4} + \cdots + t^2 + 1.$$

Choose any positive number τ from $(\frac{n-2}{m}\pi, \frac{n}{m}\pi)$. Then

$$p_\tau(t) = \prod_{j=\frac{n}{2}}^{\frac{m}{2}-1} (1 - \frac{t}{\omega^j})(1 - \frac{t}{\bar{\omega}^j}) = \prod_{j=\frac{n}{2}}^{\frac{m}{2}-1} (t - \omega^j)(t - \bar{\omega}^j).$$

By Theorem D, $p_\tau(t)$ is a polynomial with positive coefficients. So $\prod_{x\in S^c}(t-x)$ equals $p_\tau(t)$ and is a polynomial with positive coefficients.

When m is even and n is odd, we take $p(t)$ to be the same polynomial, but choose τ from $(\frac{n-1}{m}\pi, \frac{n+1}{m}\pi)$. A little calculation shows that in this case $\prod_{x\in S^c}(t-x)$ is equal to $(t+1)p_\tau(t)$ and so it also has positive coefficients.

When m, n are both odd, we take $p(t)$ to be the polynomial $t^{m-1}+t^{m-2}+\cdots+1$ and apply a similar argument.

The proof for Theorem 7.1 is complete. ■

Conjecture 7.4. For any positive integers $m, n, 3 \le n \le m$,

$$\max\{\gamma(K) : K \in \mathcal{P}(m,n)\} = (n-1)(m-1),$$

when m is odd and n is even.

By Theorem 5.3(I) the Conjecture is confirmed for the minimal cone case.

8. Uniqueness of exp-maximal cones and their exp-maximal primitive matrices

Given positive integers m, n with $3 \le n \le m$, up to linear isomorphism, how many exp-maximal cones are there in $\mathcal{P}(m,n)$? For a given exp-maximal cone K in $\mathcal{P}(m,n)$, up to cone-equivalence modulo positive scalar multiplication, how many exp-maximal K-primitive matrices are there ? In this section we are going to address these questions for the cases $m = n, m = n+1$ and $n = 3$.

Since there is, up to linear isomorphism, only one simplicial cone of a given dimension, we need not treat the problem of identifying exp-maximal cones in $\mathcal{P}(m,n)$ for the case $m = n$. The problem of identifying exp-maximal minimal cones has already been carried out in Section 5. According to Theorem 5.3, a cone $K \in \mathcal{P}(n+1,n)$ is exp-maximal if and only if K is an indecomposable minimal cone with a balanced relation for its extreme vectors or n is even and K is the direct sum of a ray and an $(n-1)$-dimensional indecomposable minimal cone with a balanced relation for its extreme vectors. But, by Theorem 2.6 for every positive integer $n \ge 3$, there is, up to linear isomorphism, only one n-dimensional indecomposable minimal cone with a balanced relation for its extreme vectors, so n-dimensional exp-maximal minimal cones are known completely: up to linear isomorphism, there are

one such cone when n is odd and two such cones when n is even. Some work on identifying exp-maximal 3-dimensional cones has also been done in Section 6. By Theorem 6.4(iii), for every positive integer $m \geq 3$, there are uncountably infinitely many exp-maximal cones K_θ in $\mathcal{P}(m, 3)$, one for each $\theta \in (\frac{2\pi}{m}, \frac{2\pi}{m-1})$. Later in this section, we will show that when $m \geq 4$, up to linear isomorphism, the K_θ's are precisely all the exp-maximal cones in $\mathcal{P}(m, 3)$ and, moreover, when $m \geq 6$, for different values of θ, the corresponding K_θ's are not linearly isomorphic. (At present, we *do not* know whether the latter assertion can be extended to cover the case $m = 5$.)

Once the exp-maximal cones in $\mathcal{P}(m, n)$ have been identified, the next natural problem to consider is to identify, for each typical exp-maximal cone K, all the exp-maximal K-primitive matrices. So we will also identify, up to cone-equivalence and scalar multiples, the exp-maximal primitive matrices for exp-maximal minimal cones, for exp-maximal 3-dimensional cones, as well as for simplicial cones. As a matter of fact, in identifying exp-maximal minimal cones in Section 5, we have already provided (implicitly in Lemma 5.1), up to cone-equivalence and scalar multiples, all the exp-maximal primitive matrices for minimal cones. Also, as we will show, for $m \geq 4$, if $K \in \mathcal{P}(m, 3)$ is an exp-maximal polyhedral cone and A is an exp-maximal K-primitive matrix then, up to scalar multiples, A is cone-equivalent to A_θ for some $\theta \in (\frac{2\pi}{m}, \frac{2\pi}{m-1})$, where A_θ is the matrix introduced in Theorem 6.4, and moreover, provided that $m \geq 6$, A_θ is (up to scalar multiples) the only exp-maximal K_θ-primitive matrix. However, there are exp-maximal primitive matrices for the cone $\mathbb{R}^3_+$ that are (up to cone-equivalence and scalar multiples) not of the form A_θ, where $\theta \in (\frac{2\pi}{3}, \pi)$.

We deal with the minimal cone case first. We begin with a result, which is true not only for the minimal cone case.

Lemma 8.1. *Let $K \in \mathcal{P}(m, n)$ be indecomposable. If $A, \tilde{A}$ are different K-nonnegative matrices such that the digraphs $(\mathcal{E}, \mathcal{P}(A, K))$ and $(\mathcal{E}, \mathcal{P}(\tilde{A}, K))$ are given both by Figure* 1 *or both by Figure* 2*, then $\tilde{A}$ and A are not cone-equivalent.*

Proof. Let A and $\tilde{A}$ be cone-equivalent K-nonnegative matrices such that the digraphs $(\mathcal{E}, \mathcal{P}(A, K))$ and $(\mathcal{E}, \mathcal{P}(\tilde{A}, K))$ are given both by Figure 1 or both by Figure 2. Then there exists an automorphism P of K such that $P\tilde{A} = AP$. We contend that for $j = 1, \ldots, m, P$ maps x_j to a positive multiple of itself. Once this is proved, it will follow that $P \in \Phi(I)$. But K is indecomposable, so by [19, Theorem 3.3] $\Phi(I)$ is an extreme ray of $\pi(K)$; hence P is a positive multiple of I and we have $\tilde{A} = A$, as desired.

To prove our contention, we first deal with the case when the digraphs $(\mathcal{E}, \mathcal{P}(A, K))$ and $(\mathcal{E}, \mathcal{P}(\tilde{A}, K))$ are both given by Figure 1. As P is an automorphism of K, P permutes the extreme rays of K among themselves. According to the proof of Lemma 4.2(ii), the maximum value of $\gamma(A, x)$, for $x = x_1, \ldots, x_m$, is attained at x_1 only. When A is replaced by $\tilde{A}$, the same can be said. Since A and $\tilde{A}$ are cone-equivalent, by Fact 2.5(v) P must map x_1 to a positive multiple of itself. Making use of the relation $P\tilde{A} = AP$ and the fact that Ax_i is a positive multiple of x_{i+1} for $i = 1, \ldots, m-1$ and proceeding inductively, we readily show that P maps each x_i to a positive multiple of itself, which is our contention.

If $(\mathcal{E}, \mathcal{P}(A, K))$ and $(\mathcal{E}, \mathcal{P}(\tilde{A}, K))$ are both given by Figure 2, we readily show that Px_2 must be a positive multiple of itself and then we can proceed in a similar manner. ■

Next, a remark on the automorphisms of an indecomposable minimal cone is in order.

Let K be an indecomposable minimal cone in $\mathbb{R}^n$ with extreme vectors $x_1, \ldots, x_{n+1}$ that satisfy the relation

$$x_1 + \cdots + x_p = x_{p+1} + \cdots + x_{n+1}.$$

Let σ be a permutation on $\{1, \ldots, n+1\}$ that maps $\{1, \ldots, p\}$ and $\{p+1, \ldots, n\}$ each onto itself, or interchanges the first set with the second set (in which case n is odd and $p = \frac{n+1}{2}$). Then σ determines a (unique) automorphism P_σ of K with spectral radius 1 ($x_1 + \cdots + x_p$ being the corresponding eigenvector) which is given by: $P_\sigma x_j = x_{\sigma(j)}$ for $j = 1, \ldots, n+1$. Conversely, every automorphism of K whose spectral radius is 1 arises in this way.

Now we consider the exp-maximal primitive matrices for indecomposable exp-maximal minimal cones first. We give two results, one for the odd dimensional case and the other for the even dimensional case. The relations (5.1), (5.2), (5.3), (5.4), (5.6), (5.7) that are mentioned in these results have already appeared in Lemma 5.1.

Theorem 8.2. *Let K be an n-dimensional indecomposable exp-maximal minimal cone, where n is odd. Suppose that the extreme vectors $x_1, \ldots, x_{n+1}$ of K satisfy relation* (5.1) *with* (*with $m = n+1$*). *For every $\alpha > 0$, let A_α be the exp-maximal K-primitive matrix given by* (5.2) (*but with A replaced by A_α*). *Then*:

(i) *$\Phi(A_\alpha)$ is a 2-dimensional face, independent of the choice of the positive scalar α; its relative interior consists of positive multiples of matrices of the form $A_{\tilde{\alpha}}$.*

(ii) *Every exp-maximal K-primitive matrix is cone-equivalent to a positive multiple of some A_α and thus is a positive multiple of a matrix of the form $P_\sigma^{-1}A_\alpha P_\sigma$, where P_σ is the automorphism of K given by $P_\sigma x_j = x_{\sigma(j)}$ for $j = 1, \ldots, n+1$, σ being a permutation on $\{1, \ldots, n+1\}$ that maps $\{1, 3, \ldots, n-2, n\}$ onto itself or onto $\{2, 4, \ldots, n-1, n+1\}$.*

(iii) *For distinct positive scalars α_1, α_2, A_{α_1} and A_{α_2} or their positive multiples are not cone-equivalent.*

Proof (i) It is clear that $\Phi(A_\alpha) = \Phi(A_{\tilde{\alpha}})$ for any $\alpha, \tilde{\alpha} > 0$ as $\Phi(A_\alpha x_j) = \Phi(A_{\tilde{\alpha}} x_j)$ for each j and K is polyhedral. We are going to show that $\Phi(A_\alpha)$ is equal to the 2-dimensional face generated by the extreme matrices B, C determined respectively by:

$$Bx_i = x_{i+1} \text{ for } i = 1, \ldots, m, \text{ where } x_{m+1} \text{ is taken to be } x_1,$$

and

$$Cx_1 = x_2 = Cx_m, Cx_i = 0 \text{ for } i = 2, \ldots, m-1.$$

It is readily checked that B and C each preserve relation (5.1); so they are well-defined and K-nonnegative. Since $A_\alpha = B + \alpha C$, we have $B, C \in \Phi(A_\alpha)$ and hence $\operatorname{pos}\{B, C\} \subseteq \Phi(A_\alpha)$. To complete the argument, we contend that every matrix in $\operatorname{ri}\Phi(A_\alpha)$ is a positive multiple of some $A_{\tilde{\alpha}}$ (and hence belongs to $\operatorname{pos}\{B, C\}$). Once this is proved, the desired reverse inclusion follows as $\Phi(A_\alpha) = \operatorname{cl}[\operatorname{ri}\Phi(A_\alpha)]$.

Consider any K-nonnegative matrix $\tilde{A}$ that satisfies $\Phi(\tilde{A}) = \Phi(A_\alpha)$. Since $\Phi(\tilde{A}x_m) = \Phi(A_\alpha x_m)$ and $\Phi(A_\alpha x_m)$ is the 2-dimensional face of K generated by x_1, x_2, after normalizing $\tilde{A}$, we may assume that $\tilde{A}x_m = x_1 + \tilde{\alpha}x_2$ for some $\tilde{\alpha} > 0$. Similarly, we may assume that $\tilde{A}x_i = \tilde{\alpha}_{i+1}x_{i+1}$ for $i = 1, \ldots, m-1$. Substituting the values of the $\tilde{A}x_i$'s into the relation obtained from (5.1) by applying $\tilde{A}$, we obtain

$$\tilde{\alpha}_2 x_2 + \tilde{\alpha}_4 x_4 + \cdots + \tilde{\alpha}_{m-2}x_{m-2} + \tilde{\alpha}_m x_m = \tilde{\alpha}_3 x_3 + \tilde{\alpha}_5 x_5 + \cdots + \tilde{\alpha}_{m-1}x_{m-1} + x_1 + \tilde{\alpha}x_2.$$

Since (5.1) is, up to multiples, the only relation for the extreme vectors of K, we have

$$\tilde{\alpha}_i = 1 = \tilde{\alpha}_2 - \tilde{\alpha} \text{ for } i = 3, \ldots, m.$$

Hence $\tilde{A}$ is given by

$$\tilde{A}x_1 = (1 + \tilde{\alpha})x_2, \tilde{A}x_i = x_{i+1} \text{ for } i = 2, \ldots, m-1, \text{ and } \tilde{A}x_m = x_1 + \tilde{\alpha}x_2,$$

for some $\tilde{\alpha} > 0$. This proves that, after normalization, $\tilde{A}$ equals some $A_{\tilde{\alpha}}$, which is our contention.

(ii) Let A be an exp-maximal K-primitive matrix. In view of Theorem 5.3(II)(ii), the digraph $(\mathcal{E}, \mathcal{P}(A, K))$ is given by Figure 1, except that $x_1, \ldots, x_m$ are to be replaced by $x_{\sigma(1)}, \ldots, x_{\sigma(m)}$ respectively, where σ is some permutation on $\{1, \ldots, m\}$. By Lemma 5.1(i) we can find positive scalars α and $\lambda_j, j = 1, \ldots, m$, such that, after normalizing A, the relation on Ext K and the matrix A are given respectively by the relations obtained from (5.1) and (5.2) by replacing each x_j by $\lambda_j x_{\sigma(j)}$. Since the relation on $x_1, \ldots, x_m$ is, up to multiples, unique, it follows that $\lambda_1, \ldots, \lambda_m$ are the same and moreover σ maps the set $\{1, 3, \ldots, n\}$ onto itself or onto $\{2, 4, \ldots, n+1\}$. It is readily checked that we have $A = P_\sigma^{-1} A_\alpha P_\sigma$. So, after normalization, A is equivalent to some A_α.

(iii) As can be readily seen, if α_1, α_2 are distinct positive scalars, then A_{α_1} and A_{α_2} are linearly independent. So by Lemma 8.1 a positive multiple of A_{α_1} cannot be cone-equivalent to a positive multiple of A_{α_2}. ∎

Theorem 8.3. *Let K be an n-dimensional indecomposable exp-maximal minimal cone, where n is even. Suppose that the extreme vectors $x_1, \ldots, x_{n+1}$ of K satisfy relation* (5.3) (*with $m = n+1$*). *For every $\alpha > 0$, let A_α be the exp-maximal K-primitive matrix given by* (5.4) (*but with A replaced by A_α*). *For every $\alpha, \beta > 0$, let $A_{\alpha,\beta}$ be the K-nonnegative matrix defined by* :

$$\begin{aligned} A_{\alpha,\beta} x_1 &= \beta x_2, \\ A_{\alpha,\beta} x_3 &= (1+\alpha) x_1 + (1+\beta) x_2, \\ A_{\alpha,\beta} x_n &= x_3 + \alpha x_1, \\ A_{\alpha,\beta} x_i &= \begin{cases} x_{i+3} & \text{when } i \text{ is even}, i \neq n \\ x_{i-1} & \text{when } i \text{ is odd}, i \neq 1, 3. \end{cases} \end{aligned}$$

Then :

(i) $\Phi(A_\alpha)$ *is a 2-dimensional face, independent of the choice of the positive scalar α; its relative interior consists of positive multiples of matrices of the form $A_{\tilde{\alpha}}$.*

(ii) $\Phi(A_{\alpha,\beta})$ *is a 3-dimensional simplicial face, independent of the choice of the positive scalars α, β; its relative interior consists of positive multiples of matrices of the form $A_{\tilde{\alpha},\tilde{\beta}}$.*

(iii) *Every exp-maximal K-primitive matrix is cone-equivalent to a positive multiple of some A_α or some $A_{\alpha,\beta}$ and thus is a positive multiple of a matrix of*

the form $P_\sigma^{-1}A_\alpha P_\sigma$ or $P_\sigma^{-1}A_{\alpha,\beta}P_\sigma$, where P_σ is the automorphism of K given by $P_\sigma x_j = x_{\sigma(j)}$ for $j = 1, \dots, n+1$, σ being a permutation on $\{1, \dots, n+1\}$ that maps the set $\{1, 2, 4, \dots, n-2, n\}$ onto itself.

(iv) *The A_α's, $A_{\alpha,\beta}$'s or their positive multiples are pairwise not cone-equivalent.*

Proof (i) Use the same argument as that for Theorem 8.2(i).

(ii) Use the same kind of argument as that for Theorem 8.2(i); in this case we can show that $\Phi(A_{\alpha,\beta})$ is the 3-dimensional simplicial face generated by the extreme matrices B, C, D given respectively by:

$$Bx_1 = 0, Bx_3 = x_1 + x_2, Bx_n = x_3, \text{ and}$$

$$Bx_i = \begin{cases} x_{i+3} & \text{when } i \text{ is even}, i \neq n \\ x_{i-1} & \text{when } i \text{ is odd}, i \neq 1, 3 \end{cases}$$

$$Cx_3 = x_1 = Cx_n, Cx_i = 0 \text{ for } i \neq 3, n$$

and

$$Dx_1 = x_2 = Dx_3, Dx_i = 0 \text{ for } i \neq 1, 3.$$

(iii) We first show that $A_{\alpha,\beta}$ is exp-maximal K-primitive. Let $\tilde{K}$ denote the n-dimensional indecomposable minimal cone with extreme vectors $x_1, \dots, x_{n+1}$ that satisfy relation (5.6) (with $m = n+1$), and let $\tilde{A}$ be the $\tilde{K}$-nonnegative matrix defined by (5.7) (but with A replaced by $\tilde{A}$). By Lemma 5.2(ii) and Theorem 5.3(I), $\tilde{A}$ is exp-maximal $\tilde{K}$-primitive. Let π be the permutation on $\{1, 2, \dots, n+1\}$ given by $\pi(1) = 3, \pi(2) = 1$, and

$$\pi(j) = \begin{cases} j-1 & \text{when } j \text{ is odd}, j \neq 1 \\ j+1 & \text{when } j \text{ is even}, j \neq 2. \end{cases}$$

Let P be the matrix from $\operatorname{span} K$ to $\operatorname{span} \tilde{K}$ given by $Px_j = x_{\pi(j)}$. Then, as can be readily checked, P is an isomorphism which maps K onto $\tilde{K}$ and, moreover, we have $A_{\alpha,\beta} = P_\pi^{-1}AP_\pi$. So $A_{\alpha,\beta}$ is cone-equivalent to $\tilde{A}$ and hence is exp-maximal K-primitive.

Let A be an exp-maximal K-primitive matrix. In view of Theorem 5.3(III)(ii), the digraph $(\mathcal{E}, \mathcal{P}(A, K))$ is given by Figure 1 or Figure 2 (with $m = n+1$), except that $x_1, \dots, x_{n+1}$ are to be replaced by $x_{\sigma(1)}, \dots, x_{\sigma(n+1)}$ respectively, where σ is some permutation on $\{1, \dots, n+1\}$. In the former case, following the argument given in the proof for Theorem 8.2(ii), we can show that A is a positive multiple

of a matrix of the form $P_\sigma^{-1}A_\alpha P_\sigma$ where P_σ is the automorphism of K given by $P_\sigma x_j = x_{\sigma(j)}$ for $j = 1, \ldots, n+1$, σ being a permutation on $\{1, \ldots, n+1\}$ that maps the set $\{1, 2, 4, \ldots, n-2, n\}$ onto itself, noting that σ cannot interchange the sets $\{1, 2, 4, \ldots, m-3, m-1\}$ and $\{3, 5, \ldots, m-2, m\}$ as their cardinality differ by 1. So, in this case, A is cone-equivalent to a positive multiple of some A_α.

When the digraph $(\mathcal{E}, \mathcal{P}(A, K))$ is given by Figure 2 (but with vertices relabelled as indicated above), by Lemma 5.1(ii) we can find positive scalars α, β and $\lambda_j, j = 1, \ldots, m$, such that, after normalizing A, the relation on $\operatorname{Ext} K$ and the matrix A are given by the relations obtained from (5.6) and (5.7) respectively by replacing each x_j by $\lambda_j x_{\sigma(j)}$. But the relation on $\operatorname{Ext} K$, which is also given by (5.3), is unique (up to multiples), it follows that all the λ_j's are the same. So the relation on $\operatorname{Ext} K$ and the matrix A are given respectively by (5.6) and (5.7), but with each x_j replaced by $x_{\sigma(j)}$. But then A is cone-equivalent to the matrix $\tilde{A}$, which was introduced at the beginning of the proof for this part, and hence is also cone-equivalent to $A_{\alpha,\beta}$, as desired.

(iv) As done in the proof for Theorem 8.3(iii), if α_1, α_2 are distinct positive scalars, then a positive multiple of A_{α_1} and a positive multiple of A_{α_2} are linearly independent, and so by Lemma 8.1 they are not cone-equivalent. For a similar reason, a positive multiple of A_{α_1,β_1} is also not cone-equivalent to a positive multiple of A_{α_2,β_2}, provided that $(\alpha_1, \beta_1) \neq (\alpha_2, \beta_2)$. Moreover, a matrix of the form A_α and one of the form $A_{\beta,\gamma}$, or their positive multiples, are also not cone-equivalent, because $(\mathcal{E}, \mathcal{P}(A_\alpha))$ is given by Figure 1 whereas $(\mathcal{E}, \mathcal{P}(A_{\beta,\gamma}))$ is given by Figure 2, and the two digraphs are not isomorphic. ■

Now we consider the exp-maximal primitive matrices for a decomposable exp-maximal minimal cone. In this case, Lemma 8.1 no longer applies. What we have is the following:

Lemma 8.4. *Let $K \in \mathcal{P}(n+1, n)$ be an exp-maximal decomposable minimal cone with extreme vectors $x_1, \ldots, x_{n+1}$ (where n is even). Suppose that $K = \operatorname{pos}\{x_2\} \oplus \operatorname{pos}\{x_1, x_3, x_4, \ldots, x_{n+1}\}$, where $x_1, x_3, x_4, \ldots, x_{n+1}$ satisfy the relation given by* (5.9) *(with $m = n+1$). Let A and $\tilde{A}$ be the K-nonnegative matrices defined respectively by* (5.10) *and by the relation obtained from* (5.10) *by replacing A, α, β by $\tilde{A}, \tilde{\alpha}, \tilde{\beta}$ respectively. Then for any $\omega > 0$, $\tilde{A}$ and ωA are cone-equivalent if and only if $\omega = 1$ and $\alpha\beta = \tilde{\alpha}\tilde{\beta}$.*

Proof. "Only if" part: First, note that the given assumptions guarantee that the digraphs $(\mathcal{E}, \mathcal{P}(A, K))$ and $(\mathcal{E}, \mathcal{P}(\tilde{A}, K))$ are both given by Figure 2 (see Lemma 5.2(iii)).

Suppose that $\tilde{A}$ and ωA are cone-equivalent. Let P be an automorphism of K such that $P\tilde{A} = \omega AP$. By the argument given in the proof of Lemma 8.1, we can show that P takes x_2 to a positive multiple of itself. Then from the relation $P\tilde{A}x_2 = \omega APx_2$ we infer that P also maps x_3 to a positive multiple of itself. Proceeding inductively, we can show that P maps each x_j to a positive multiple of itself. Say, we have $Px_i = \lambda_i x_i$ for $i = 1, \ldots, m$. Substituting the values of the Px_i's into the relation obtained from (5.9) by applying P and using the fact that, up to multiples, (5.9) is the only relation for the extreme vectors of K, we conclude that all the λ_j's, for $j = 1, \ldots, m, j \neq 2$, are equal. Denote their common value by λ. Then P is given by $Px_i = \lambda x_i$ for $i = 1, \ldots, m, i \neq 2$, and $Px_2 = \mu x_2$, where μ denotes λ_2. Now by the given assumptions on A and $\tilde{A}$ we have

$$P\tilde{A}x_1 = P(\tilde{\alpha}x_2 + x_3) = \tilde{\alpha}\mu x_2 + \lambda x_3 \text{ and } \omega APx_1 = \omega A(\lambda x_1) = \omega\lambda(\alpha x_2 + x_3).$$

But $P\tilde{A}x_1 = \omega APx_1$, so we obtain $\omega = 1$ and $\tilde{\alpha}/\alpha = \lambda/\mu$. Then the relation $P\tilde{A} = \omega AP$ reduces to $P\tilde{A} = AP$. Similarly, from the relation $P\tilde{A}x_2 = APx_2$ we obtain $\beta/\tilde{\beta} = \lambda/\mu$. Hence we have $\alpha\beta = \tilde{\alpha}\tilde{\beta}$.

Conversely, suppose that $\alpha\beta = \tilde{\alpha}\tilde{\beta}$. Choose positive scalars λ, μ such that $\lambda/\mu = \tilde{\alpha}/\alpha (= \beta/\tilde{\beta})$. Let P be the automorphism of K determined by $Px_2 = \mu x_2$ and $Px_i = \lambda x_i$ for all $i \neq 2$. Then, as can be readily checked, $P\tilde{A}x_i = APx_i$ for every i. Hence we have $P\tilde{A} = AP$, i.e., A and $\tilde{A}$ are cone-equivalent. ■

In view of Lemma 8.4 and using the kind of argument as given in the proofs for Theorem 8.2 and Theorem 8.3, we can establish the following, whose proof we omit:

Theorem 8.5. *Let $K \in \mathcal{P}(n+1, n)$ be an exp-maximal decomposable minimal cone with extreme vectors $x_1, \ldots, x_{n+1}$ (where n is even). Suppose that $K = \text{pos}\{x_2\} \oplus \text{pos}\{x_1, x_3, x_4, \ldots, x_{n+1}\}$, where $x_1, x_3, x_4, \ldots, x_{n+1}$ satisfy the relation given by* (5.9) *(with $m = n+1$). For every $\alpha, \beta > 0$, let $A_{\alpha,\beta}$ be the exp-maximal K-primitive matrix defined by* (5.10) *(but with A replaced by $A_{\alpha,\beta}$), and assume that $x_1, \ldots, x_m$ satisfy relation* (5.9)*. Then :*

(i) $\Phi(A_{\alpha,\beta})$ *is a* 3*-dimensional simplicial face, independent of the choice of the positive scalars α, β; its relative interior consists of positive multiples of matrices of the form $A_{\tilde{\alpha},\tilde{\beta}}$.*

(ii) *Every exp-maximal K-primitive matrix is cone-equivalent to a positive multiple of some $A_{1,\beta}$.*

(iii) *For distinct positive scalars β_1, β_2, the matrices $A_{1,\beta_1}, A_{1,\beta_2}$, or their positive multiples, are pairwise not cone-equivalent.*

In view of the preceding theorems, we can conclude that for every exp-maximal minimal cone K, indecomposable or not, there are uncountably infinitely many exp-maximal K-primitive matrices which are pairwise linearly independent and non-cone-equivalent.

Next, we consider the 3-dimensional cone case. We will need the following known result ([13, Lemma 5.3]):

Theorem E. *Let $\{x_1, \ldots, x_n\}$ be a basis for $\mathbb{R}^n$. Let $x_0 = \sum_{i=1}^n \alpha_i x_i$ where each α_i is different from 0. Let A and B are $n \times n$ real matrices. Suppose that A is nonsingular and also that Bx_j is a multiple of Ax_j for $j = 0, 1, \ldots, n$. Then B is a multiple of A.*

Theorem 8.6. *Let $m \geq 4$ be a positive integer. For each $\theta \in (\frac{2\pi}{m}, \frac{2\pi}{m-1})$, let K_θ and A_θ be respectively the exp-maximal cone and exp-maximal K_θ-primitive matrix as defined in Theorem 6.4*(iii).

(i) *If $K \in \mathcal{P}(m, 3)$ is an exp-maximal polyhedral cone and A is an exp-maximal K-primitive matrix with $\rho(A) = 1$, then there exists a unique $\theta \in (\frac{2\pi}{m}, \frac{2\pi}{m-1})$ such that A is cone-equivalent to A_θ.*

(ii) *When $m \geq 6, A_\theta$ is, up to positive scalar multiples, the only exp-maximal K_θ-primitive matrix.*

Proof. (i) Since A is exp-maximal, by Theorem 4.4(i), after relabelling its vertices, we may assume that the digraph $(\mathcal{E}, \mathcal{P}(A, K))$ is given by Figure 1. Let $v \in \operatorname{int} K^*$ be a Perron vector of A^T and denote by C the complete cross-section $\{x \in K : \langle x, v \rangle = 1\}$ of K, which is a polygon with m extreme points. By the proof of Lemma 4.2(i) and in view of Lemma 6.1, we may assume that $x_1, \ldots, x_m$ are precisely all the extreme points of C and x_i, x_{i+1} are neighborly extreme points for $i = 1, \ldots, m$ (where x_{m+1} is taken to be x_1); also, $Ax_j = x_{j+1}$ for $j = 1, \ldots, m-1$ and $Ax_m = (1-c)x_1 + cx_2$ for some $c \in (0, 1)$, and moreover $t^m - ct - (1-c)$ is an annihilating polynomial for A. Let u be the Perron vector of A that belongs to C and denote by $\hat{C}$ the polygon $C - u$. Note that $\operatorname{span} \hat{C}$ equals $(\operatorname{span}\{v\})^\perp$ and so it is invariant under A (as v is an eigenvector of A^T). Let λ_1, λ_2 denote the eigenvalues of the restriction of A to $(\operatorname{span}\{v\})^\perp$. It is clear that the eigenvalues of A are 1 (the Perron root) and λ_1, λ_2. By the Perron-Frobenius theory, $|\lambda_j| < 1$ for $j = 1, 2$. We contend that λ_1, λ_2 form a conjugate pair of non-real complex numbers.

For $j = 1, \ldots, m$, denote by y_j the point $x_j - u$. Clearly, $y_1, \ldots, y_m$ are all the extreme points of $\hat{C}$ and y_i, y_{i+1} are neighborly extreme points for $i = 1, \ldots, m$

(where y_{m+1} is taken to be y_1); and $0 \in \mathrm{ri}\,\hat{C}$ as $u \in \mathrm{ri}\,C$. Since $Au = u$, the action of A on C induces a corresponding action on $\hat{C}$: we have $Ay_j = y_{j+1}$ for $j = 1, \ldots, m-1$, and $Ay_m = (1-c)y_1 + cy_2$. Take note of the following consequences of the fact that the digraph $(\mathcal{E}, \mathcal{P}(A,K))$ is given by Figure 1: A maps no extreme points of C into $\mathrm{ri}\,C$ and maps the relative interior of precisely one 2-dimensional face of C into $\mathrm{ri}\,C$. The preceding assertion is still true if C is replaced by $\hat{C}$.

Since $t^m - ct - (1-c)$ is an annihilating polynomial for A, we have $\lambda_j^m = c\lambda_j + (1-c)\cdot 1$. Suppose that λ_1, λ_2 are real. Then clearly we have $\lambda_1, \lambda_2 \in (-1, 0)$. Let $w \in (\mathrm{span}\{v\})^\perp$ be an eigenvector of A corresponding to λ_1 and suppose that $\mathrm{span}\{w\}$ meets the relative boundary of $\hat{C}$ at the points z_1, z_2. Then z_1, z_2 are scalar multiples of w but with opposite signs; say, we have $z_1 = a_1 w$ and $z_2 = a_2 w$ with $|a_2| \geq |a_1|$. A little calculation shows that $Az_1 = \alpha z_2$ for some $\alpha \in (0,1)$; as $0 \in \mathrm{ri}\,\hat{C}$, this implies that $Az_1 \in \mathrm{ri}\,\hat{C}$. If z_1 is an extreme point of $\hat{C}$, we already obtain a contradiction, as A sends no extreme point of $\hat{C}$ to $\mathrm{ri}\,\hat{C}$. So suppose that z_1 lies in the relative interior of a side of the polygon $\hat{C}$. Now the point Az_2, which is a positive multiple of z_1, either lies in $\mathrm{int}\,\hat{C}$ or is equal to z_1. Suppose $Az_2 \in \mathrm{int}\,\hat{C}$. Since A sends no extreme point of $\hat{C}$ to $\mathrm{int}\,\hat{C}$, it follows that z_2 is not an extreme point and so it lies in the relative interior of a side of the polygon $\hat{C}$. But then A maps the relative interior of two different sides of $\hat{C}$ into its relative interior, which is a contradiction. So we must have $Az_2 = z_1$. If z_2 is an extreme point, then necessarily $z_2 = y_m$ and the side of $\hat{C}$ that contains z_1 is the line segment $\overline{y_1 y_2}$. On the other hand, since $Ay_1 = y_2$ and $Ay_2 = y_3$, we obtain $Az_1 \in \mathrm{ri}\,\overline{y_2 y_3}$, which contradicts the fact that $Az_1 \in \mathrm{ri}\,\hat{C}$. So z_2 must lie in the relative interior of a side of $\hat{C}$. Then necessarily the side of $\hat{C}$ that contains z_2 is $\overline{y_{m-2} y_{m-1}}$, whereas the side that contains z_1 is $\overline{y_{m-1} y_m}$. Since $z_2 \in \mathrm{ri}\,\overline{y_{m-2} y_{m-1}}$, we have $z_1 = Az_2 \in \mathrm{ri}\,\overline{y_{m-1} y_m}$ and hence $Az_1 \in \mathrm{ri}\,\mathrm{conv}\{y_m, y_1, y_2\}$. Note that Az_1 also belongs to $\mathrm{ri}\,\mathrm{conv}\{y_{m-2}, y_{m-1}, y_m\}$ as it lies in $\mathrm{ri}\,\overline{z_1 z_2}$ and $z_1 \in \mathrm{ri}\,\overline{y_{m-1} y_m}$, $z_2 \in \mathrm{ri}\,\overline{y_{m-2} y_{m-1}}$. But $\mathrm{ri}\,\mathrm{conv}\{y_m, y_1, y_2\} \cap \mathrm{ri}\,\mathrm{conv}\{y_{m-2}, y_{m-1}, y_m\} = \emptyset$ as $m \geq 4$, so we arrive at a contradiction.

In the above we have shown that the eigenvalues λ_1, λ_2 of A form a conjugate pair of non-real complex numbers, say, they are $\pm re^{i\theta}$. Then clearly $r < 1$. We can now choose a basis $\{u_1, u_2\}$ for $(\mathrm{span}\{v\})^\perp$ with $u_1 = y_1$ such that

$$\begin{aligned} Au_1 &= r\cos\theta\; u_1 + r\sin\theta\; u_2, \\ Au_2 &= -r\sin\theta\; u_1 + r\cos\theta\; u_2. \end{aligned}$$

For $j = 1, \dots, m$, let $y_j = \alpha_j u_1 + \beta_j u_2$. Then we have

$$\begin{pmatrix} \alpha_{j+1} \\ \beta_{j+1} \end{pmatrix} = r \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \alpha_j \\ \beta_j \end{pmatrix}$$

for $j = 1, \dots, m-1$, as $Ay_j = y_{j+1}$ for every such j. Since $y_1, \dots, y_m$ form the consecutive vertices of a polygon in $(\mathrm{span}\{v\})^\perp$, the points $(\alpha_j, \beta_j)^T, j = 1, \dots, m$, also form the consecutive vertices of a polygon in $\mathbb{R}^2$. Now it should be clear that we have $(m-1)\theta < 2\pi$ and $2\pi < m\theta$, which implies that $\theta \in (\frac{2\pi}{m}, \frac{2\pi}{m-1})$.

As $(\alpha_1, \beta_1)^T = (1, 0)^T$, a little calculation gives $Ay_m = r^m \cos m\theta u_1 + r^m \sin m\theta u_2$. Since u_1, u_2 are linearly independent, the relation $Ay_m = (1-c)y_1 + cy_2$ leads to relations (6.1) and (6.2) that appear in the proof of Lemma 6.2. Eliminating c from these two relations, we obtain

$$\frac{\sin(m-1)\theta}{\sin\theta} r^m - \frac{\sin m\theta}{\sin\theta} r^{m-1} + 1 = 0;$$

hence r equals r_θ, the unique positive real root of the polynomial $g_\theta(t)$. It is clear that θ is unique, because we have $c = r_\theta^{m-1} \frac{\sin m\theta}{\sin\theta}$ (see Corollary 6.3(ii)). Now let P be the 3×3 matrix given by: $Pu_j = e_j$, where $u_3 = u$, the Perron vector of A, and e_j is the jth standard unit vector of $\mathbb{R}^3$. It is readily checked that P is a nonsingular matrix that maps K onto K_θ. Moreover, we have $PA = A_\theta P$. So the cones K and K_θ are linearly isomorphic, and the cone-preserving maps A and A_θ are cone-equivalent.

(ii) Let B be an exp-maximal K_θ-primitive matrix. Then $\gamma(B) = 2m+1$ and by Theorem 4.4(i) the digraph $(\mathcal{E}, \mathcal{P}(B, K_\theta))$ is, apart from the labelling of its vertices, given by Figure 1. For simplicity, we denote $x_j(\theta)$ by x_j for $j = 1, \dots, m$. Also, we adopt the convention that for any integer $j \notin \{1, \dots, m\}, x_j$ is taken to be x_k, where k is the unique integer that satisfies $1 \le k \le m, k \equiv j (\mathrm{mod}\, m)$. According to Lemma 6.1, adjacent vertices of the digraph $(\mathcal{E}, \mathcal{P}(B, K_\theta))$ correspond to neighboring extreme rays of K_θ. Using an argument similar to the one given in the proof of Theorem 6.5, we can show that there exists $p, 1 \le p \le m$ such that one of the following holds:

(I) Bx_j is a positive multiple of x_{j+1} for $j = p, p+1, \dots, p+m-2$ and Bx_{p+m-1} is a positive linear combination of x_{p+m} and x_{p+m+1}; or

(II) Bx_j is a positive multiple of x_{j-1} for $j = p, p-1, \dots, p-m+2$ and Bx_{p-m+1} is a positive linear combination of x_{p-m} and x_{p-m-1}.

We consider the case when (I) holds first. Then Bx_j is a positive multiple of $A_\theta x_j$ for all $j = 1, \dots, m$ except for $j \equiv 1, p+m-1 (\mathrm{mod}\, m)$. Since there are $m-2$

such x_j's and $m-2 \geq 4$ (as $m \geq 6$), by Theorem E, B must be a positive multiple of A_θ, which is what we want. (In fact, then necessarily $p = 1$.)

When (II) holds, we find that Bx_j is a positive multiple of $A_\theta^{-1}x_j$ for all $j = 1, \ldots, m$, except for $j \equiv 1, p-m+1 (\mathrm{mod}\, m)$. By Theorem E again we conclude that B is a positive multiple of A_θ^{-1}, which is impossible, as A_θ^{-1} is not K_θ-nonnegative. ■

Corollary 8.7. *Let $m \geq 6$ be a positive integer, and let $K_\theta \in \mathcal{P}(m, 3)$ be the exp-maximal cone as defined before. Then:*

(i) *The automorphism group of K_θ consists of scalar matrices only.*

(ii) *For any $\theta_1, \theta_2 \in (\frac{2\pi}{m}, \frac{2\pi}{m-1}), \theta_1 \neq \theta_2$, the cones $K_{\theta_1}, K_{\theta_2}$ are not linearly isomorphic.*

Proof. (i) We first establish the following:

Assertion. Every automorphism of K_θ that commutes with A_θ is a scalar matrix.

Proof. Let P be an automorphism of K_θ that commutes with A. For simplicity, denote $x_j(\theta)$ by x_j for $j = 1, \ldots, m$. Since P is an automorphism of K_θ, P permutes the extreme rays of K_θ among themselves. So Px_1 is a positive multiple of x_p for some $p \in \langle m \rangle$. In view of the relation $A_\theta P x_1 = P A_\theta x_1$, we see that Px_2 is a positive multiple of $A_\theta x_p$. But $A_\theta x_p$ is a positive multiple of x_{p+1} if $p < m$ and is a positive linear combination of x_1 and x_2 if $p = m$, so we must have $1 \leq p < m$ and Px_2 is a positive multiple of x_{p+1}. Suppose that $2 \leq p$. By considering the relation $A_\theta P x = P A_\theta x$ for $x = x_2, \ldots, m-p+1$, and in this order, and proceeding inductively, we find that $Px_j = x_{p+j-1}$ for $j = 1, \ldots, m+1-p$, and in particular we have $Px_{m-p+1} = x_m$. Then from the relation $A_\theta P x_{m-p+1} = P A_\theta x_{m-p+1}$ we infer that Px_{m+2-p} is a positive linear combination of x_1 and x_2, which is a contradiction. Thus, we have $p = 1$ and by the same kind of argument we can then show that Px_j is a positive multiple of x_j for $j = 1, \ldots, m$. Hence $P \in \Phi(I)$. Then we can invoke [19, Theorem 3.3] to conclude that $\Phi(I)$ is an extreme ray of $\Phi(I)$, and so P is a scalar matrix.

Now back to the proof of (i). Let P be an automorphism of K_θ. Since A_θ is an exp-maximal K_θ-primitive matrix, so is $P^{-1}A_\theta P$. By Theorem 8.6(ii) we have $P^{-1}A_\theta P = \alpha A_\theta$ for some $\alpha > 0$. As $P^{-1}A_\theta P$ and A_θ are similar and A_θ is nonsingular, necessarily $\alpha = 1$. So we have $A_\theta P = P A_\theta$, and by the above Assertion it follows that P is a scalar matrix.

(ii) Let $\theta_1, \theta_2 \in (\frac{2\pi}{m}, \frac{2\pi}{m-1})$ be such that the cones $K_{\theta_1}, K_{\theta_2}$ are linearly isomorphic, say, P is a linear isomorphism that maps K_{θ_2} onto K_{θ_1}. Since A_{θ_1} is exp-maximal K_{θ_1}-primitive, clearly $P^{-1}A_{\theta_1}P$, which is cone-equivalent to A_{θ_1}, is exp-maximal K_{θ_2}-primitive. In view of Theorem 8.6(ii) and the fact $\rho(A_{\theta_j}) = 1$ for $j = 1, 2$, we have $P^{-1}A_{\theta_1}P = A_{\theta_2}$. Now for $j = 1, 2$, the eigenvalues of A_{θ_j} are 1 and $r_{\theta_j}e^{\pm i\theta_j}$. So we must have $\theta_1 = \theta_2$. ∎

Note that part (i), and hence also part (ii), of Theorem 8.6 is not true for $m = 3$. This is because every A_θ has a pair of conjugate non-real complex eigenvalues, whereas an exp-maximal $\mathbb{R}^3_+$-primitive matrix need not have non-real eigenvalues (see Remark 8.9 below).

As can be readily checked, parts (i) and (ii) of Corollary 8.7 are also both invalid when $m = 3$ or 4.

We *do not* know whether Theorem 8.6(ii) or Corollary 8.7(i),(ii) can be extended to cover the case $m = 5$. However, we are going to show that the two problems are equivalent.

Upon close examination, one finds that the Assertion given in the proof of Corollary 8.7, in fact, holds also for every positive integer $m \geq 4$. Furthermore, if Theorem 8.6(ii) is true for $m = 5$, then the arguments for part (i) and (ii) of the corollary still work for the case $m = 5$.

Now suppose that Corollary 8.7 is extendable to the case $m = 5$, but Theorem 8.6(ii) is not. Then for some $\theta \in (\frac{2\pi}{5}, 2\frac{\pi}{4})$ there exists an exp-maximal K_θ-primitive matrix A, different from A_θ, such that $\rho(A) = 1$. By Theorem 8.6(i), there exists $\phi \in (\frac{2\pi}{5}, \frac{2\pi}{4})$ such that K_θ is cone-equivalent to K_ϕ and A is cone-equivalent to K_ϕ. So there exists an isomorphism P such that $PK_\phi = K_\theta$ and $A_\phi = P^{-1}AP$. In view of Corollary 8.7(ii), necessarily $\phi = \theta$. Hence, P is an automorphism of K_θ and by Corollary 8.7(i), P is a scalar matrix. Therefore, $A = A_\phi = A_\theta$, which is a contradiction.

As expected, we have the following result, which describes all the exp-maximal K-primitive matrices for K in $\mathcal{P}(n, n), n \geq 3$:

Remark 8.8. Let $K \in \mathcal{P}(n, n), n \geq 3$ and let A be a K-primitive matrix with $\rho(A) = 1$. Then A is exp-maximal K-primitive if and only if there exists $c \in (0, 1)$ such that A is cone-equivalent to $C_h(\in \pi(\mathbb{R}^n_+))$, the companion matrix of

the polynomial $h(t) = t^n - ct - (1-c)$, i.e.,

$$C_h = \begin{bmatrix} 0 & & & & 1-c \\ 1 & 0 & & \mathbf{0} & c \\ & 1 & \ddots & & 0 \\ & \mathbf{0} & \ddots & \ddots & \vdots \\ & & & 1 & 0 \end{bmatrix}_{n\times n}.$$

Proof. The "if" part is obvious as C_h is exp-maximal $\mathbb{R}^n_+$-primitive. To show the "only if" part, we may assume that $K = \mathbb{R}^n_+$. Then there exists a permutation matrix P such that $P^{-1}AP = B$, where B is a nonnegative matrix with the same zero-nonzero pattern as the companion matrix C_h. It is not difficult to find a diagonal matrix D with positive diagonal entries such that $D^{-1}BD = C_h$ for some $c \in (0,1)$. But PD is an automorphism of $\mathbb{R}^n_+$, so it follows that A is cone-equivalent to C_h. ∎

For completeness, let us mention the following result, which is not difficult to prove:

Remark 8.9. Let $K \in \mathcal{P}(3,3)$ and let A be an a K-primitive matrix with $\rho(A) = 1$. Then A is exp-maximal K-primitive if and only if A is cone-equivalent to one of the following:

(i) $A_\theta \in \pi(K_\theta)$, where $\theta \in (\frac{2\pi}{3}, \pi)$, and $K_\theta \in \mathcal{P}(3,3)$ and $A_\theta \in \pi(K_\theta)$ are defined in the same way as before;

(ii) $\mathrm{diag}(\alpha_1, \alpha_2, 1) \in \pi(\tilde{K})$, where for some $c \in (\frac{3}{4}, 1)$, α_1, α_2 are the (distinct) real roots, other than 1, of the polynomial $t^3 - ct - (1-c)$ and $\tilde{K}$ is the polyhedral cone in $\mathbb{R}^3$ generated by the extreme vectors $x_1 = (1,1,1)^T$, $x_2 = (\alpha_1, \alpha_2, 1)^T$ and $x_3 = (\alpha_1^2, \alpha_2^2, 1)^T$;

(iii) $\begin{bmatrix} -\frac{1}{2} & 1 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \in \pi(K_0)$, where $K_0 = \mathrm{pos}\{x_1, Ax_1, A^2x_1\}$ with $x_1 = (1,1,1)^T$.

9. Remarks and open questions

A positive integer κ is called the *critical exponent* of a normed space E (or of the norm on E) if the inequalities $\|A^k\| = \|A\| = 1$ imply that $\rho(A) = 1$, and if κ is the smallest number with the indicated property. It is known that not every norm in a finite-dimensional space has a critical exponent. An example of one such norm can be found in [6]. Borrowing the latter example, we are going to show that there exists a proper cone which does not have finite exponent.

Example 9.1. Let $\|\cdot\|$ denote the norm of $\mathbb{R}^2$ whose unit closed ball is defined by the inequalities:

$$3\xi_1 - 2 \leq \xi_2 \leq \xi_1^3, \text{ if } -2 \leq \xi_1 \leq -1,$$

$$3\xi_1 - 2 \leq \xi_2 \leq 3\xi_1 + 2, \text{ if } |\xi_1| \leq 1,$$

and

$$\xi_1^3 \leq \xi_2 \leq 3\xi_1 + 2, \text{ if } 1 \leq \xi_1 \leq 2.$$

(See Figure 6.) Let K be the proper cone in $\mathbb{R}^3$ given by: $K = \{\alpha\binom{x}{1} : \alpha \geq 0$ and $\|x\| \leq 1\}$. For every positive integer k, let B_k denote the 2×2 diagonal matrix $\mathrm{diag}(2^{-1/k}, 2^{-3/k})$. As shown in [6, p.67], B_k has the property that $\|B_k\| = \|B_k^k\| = 1$ but $\|B_k^{k+1}\| < 1$. Let $A_k = B_k \oplus (1)$. Then it is easy to see that A_k is K-primitive and $\gamma(A_k) = k + 1$. Since k can be arbitrarily large, this shows that for this K we have $\gamma(K) = \infty$. It is also of interest to note that the K-primitive matrices A_k obtained in this example are, in fact, all extreme matrices of the cone $\pi(K)$. The point is, K is an indecomposable cone and each of the A_k's maps infinitely many extreme rays of K onto extreme rays.

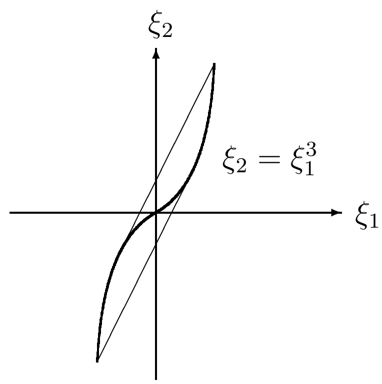

Figure 6.

Let E_n denote the set of values attained by the exponents of primitive matrices of order n. Dulmage and Mendelsohn [10] have found intervals in the set $\{1, 2, \ldots, (n-1)^2+1\}$ containing no integer which is the exponent of a primitive matrix of order n. These intervals have been called *gaps* in E_n. The problem of determining E_n or the gaps is an intricate problem, but it has been completely resolved. (See, for instance, [8].)

For a given polyhedral cone (or a proper cone) K, we can consider a similar problem — to determine the set of values attained by the exponents of K-primitive matrices. We expect that for every polyhedral cone K of dimension greater than 2 there are gaps in the set of values attained by the exponents of K-primitive matrices (but at present we *do not* have a proof for this claim). As an illustration, consider an indecomposable minimal cone $K \in \mathcal{P}(m,n)$ with a balanced relation for its extreme vectors, where n is an odd integer ≥ 5. For a K-primitive matrix A, if the digraph $(\mathcal{E}, \mathcal{P})$ is given by Figure 1 or Figure 2 then $\gamma(A)$ equals n^2-n+1 or n^2-n (see Theorem 5.3(II)) and Lemma 5.2(ii)). On the other hand, if the digraph is not given by Figure 1 or Figure 2, then by Lemma 4.1 the length of the shortest circuit in $(\mathcal{E}, \mathcal{P})$ is at most $n-1(=m-2)$ and by Remark 4.6 it follows that $\gamma(A) \leq (n-1)^2+2$. So in this case any integer lying in the closed interval $[n^2-2n+4, n^2-n-1]$ cannot be attained as the exponent of some K-primitive matrix.

Perhaps, a less difficult problem is the following:

Question 9.1. Let $m \geq 4$ be a positive integer. Determine the set of integers that can be attained as the exponent of a K-primitive matrix for some n-dimensional polyhedral cone K with m extreme rays, where $3 \leq n \leq m$.

Let $K \in \mathcal{P}(m,n)$ be an exp-maximal non-simplicial polyhedral cone and let A be an exp-maximal K-primitive matrix. According to Theorem 7.1, $\gamma(A)$ equals $(n-1)(m-1)+1$ or $(n-1)(m-1)$. In view of Theorem 4.4(i), in either case we have $n = m_A$. If $\gamma(A) = (n-1)(m-1)+1$, then by Theorem 4.4(i) again the digraph $(\mathcal{E}, \mathcal{P})$ is given by Figure 1. If $\gamma(A) = (n-1)(m-1)$, then by Theorem 4.4(ii) either $(\mathcal{E}, \mathcal{P})$ is given by Figure 1 or Figure 2, or $m_A = 3$. The last possibility cannot happen, because then we have $\gamma(A) = 2(m-1)$ and $n = m_A = 3$, in contradiction with Theorem 6.4(i). If $(\mathcal{E}, \mathcal{P})$ is given by Figure 1, then by Lemma 4.2(iii) K is indecomposable. If $(\mathcal{E}, \mathcal{P})$ is given by Figure 2, then again by Lemma 4.2(iii), K is either indecomposable or is an even-dimensional minimal cone which is the

direct sum of a ray and an indecomposable minimal cone with a balanced relation for its extreme vectors. Conversely, if K is an even-dimensional minimal cone with the said property, then by Theorem 5.3(III)(i) K is exp-maximal. We have thus characterized all decomposable non-simplicial exp-maximal polyhedral cones.

Question 9.2. Identify indecomposable exp-maximal cones in $\mathcal{P}(m,n)$ for $m > n$, $m \neq n+1$ and $n \neq 3$.

In this work we are able to identify the exp-maximal cones and the corresponding exp-maximal primitive matrices only for the extreme cases $m = n, m = n+1$ and $n = 3$. To deal with the other cases, one may consider the following question first:

Question 9.3. Given positive integers m, n with $3 \leq n \leq m$, characterize the $n \times n$ real matrices A with the property that there exists $K \in \mathcal{P}(m,n)$ such that A is K-nonnegative and $(\mathcal{E}, \mathcal{P}(A,K))$ is given by Figure 1.

In below we provide the answers to the preceding question for two special cases, namely, $n = 3, m \geq 4$, and $m = n$. For convenience, we normalize the matrices under consideration.

Remark 9.1. Let A be a 3×3 real matrix with $\rho(A) = 1$, and let $m \geq 4$ be a given positive integer. Then there exists $K \in \mathcal{P}(m,3)$ such that A is K-nonnegative and $(\mathcal{E}, \mathcal{P}(A,K))$ is given by Figure 1 (and hence A is exp-maximal K-primitive) if and only if the eigenvalues of A are $1, r_\theta e^{\pm i\theta}$, where $\theta \in (\frac{2\pi}{m}, \frac{2\pi}{m-1})$, r_θ is the unique positive real root of the polynomial $g_\theta(t)$ given in Lemma 6.2.

Proof. "Only if" part: Since $(\mathcal{E}, \mathcal{P}(A,K))$ is isomorphic to Figure 1, by Lemma 6.1(ii) and Theorem 6.4(i), A is exp-maximal K-primitive. By Theorem 8.6(i) A is cone-equivalent to A_θ for some $\theta \in (\frac{2\pi}{m}, \frac{2\pi}{m-1})$. Hence, the eigenvalues of A are the same as those for A_θ, i.e., $1, r_\theta e^{\pm i\theta}$.

"If" part: Since A and A_θ are similar, both being similar to the diagonal matrix $\mathrm{diag}(1, r_\theta e^{i\theta}, r_\theta e^{-i\theta})$, there exists a 3×3 real nonsingular matrix P such that $P^{-1}AP = A_\theta$. Take $K = PK_\theta$. As can be readily checked, A is K-nonnegative and A is cone-equivalent to A_θ. Hence $(\mathcal{E}, \mathcal{P}(A,K))$ is isomorphic to Figure 1. ∎

Remark 9.2. Let A be an $n \times n$ real matrix with $\rho(A) = 1$. Then there exists $K \in \mathcal{P}(n,n)$ such that A is K-nonnegative and the digraph $(\mathcal{E}, \mathcal{P}(A,K))$ is given

by Figure 1 (and hence A is exp-maximal K-primitive) if and only if A is non-derogatory and the characteristic polynomial of A is of the form $t^n - ct - (1-c)$, where $c \in (0,1)$.

Proof. "Only if" part: Suppose that there exists $K \in \mathcal{P}(n,n)$ such that A is K-nonnegative and the digraph $(\mathcal{E}, \mathcal{P}(A,K))$ is given by Figure 1. By Lemma 4.2 A is non-derogatory and has an annihilating polynomial of the form of the form $t^n - ct - (1-c)$, where $c \in (0,1)$. Since A is $n \times n$, clearly $t^n - ct - (1-c)$ is also the characteristic polynomial of A.

"If" part: In this case, A is similar to C_h, the companion matrix of $h(t)$. So there exists an $n \times n$ real matrix P such that $P^{-1}AP = C_h$. Let $K = P\mathbb{R}^n_+$. Then A is K-nonnegative and is cone-equivalent to C_h. By Theorem 8.8 A is exp-maximal K-primitive. But the cone K is simplicial, so the digraph $(\mathcal{E}, \mathcal{P}(A,K))$ is isomorphic to Figure 1. ■

We would like to add that in the "if" part of Remark 9.2, except for the case $c = c_n, n$ being odd, we may omit the assumption that A be non-derogatory. The point is, then by Lemma 6.2(i) A has simple eigenvalues and hence is, necessarily, non-derogatory. As an illustration for the exceptional case, consider the matrix $A = \mathrm{diag}(-\frac{1}{2}, -\frac{1}{2}, 1)$. Its characteristic polynomial is $t^3 - \frac{3}{4}t - \frac{1}{4}$, which is of the said form. (Recall that $c_3 = \frac{3}{4}$.) Since A is not non-derogatory, by Lemma 4.2(i), there cannot exist $K \in \mathcal{P}(n,n)$ such that A is K-nonnegative and the digraph $(\mathcal{E}, \mathcal{P}(A,K))$ is given by Figure 1.

Note that in Question 9.3 we do not pose the same question for Figure 2, because the two questions are equivalent; that is, the existence of a pair (K, A) with $(\mathcal{E}, \mathcal{P}(A,K))$ given by Figure 1 guarantees the existence of a pair (K, A) with $(\mathcal{E}, \mathcal{P}(A,K))$ given by Figure 2, and vice versa.

To see this, suppose that A is K-nonnegative and $(\mathcal{E}, \mathcal{P}(A,K))$ is given by Figure 2. Then $\Phi(x_1 + x_2)$ is a 2-dimensional face of K and Ax_m lies in its relative interior. Let $\hat{K}$ denote the polyhedral cone generated by $Ax_m, x_2, x_3, \ldots, x_m$. It is readily shown that $Ax_m, x_2, \ldots, x_m$ are precisely (up to multiples) all the extreme vectors of $\hat{K}$. Furthermore, A is $\hat{K}$-nonnegative and $(\mathcal{E}, \mathcal{P}(A, \hat{K}))$ is isomorphic to Figure 1 (under the isomorphism given by: $\Phi(Ax_m) \mapsto \Phi(x_m), \Phi(x_j) \mapsto \Phi(x_{j-1})$ for $j = 2, \ldots, m$). Conversely, suppose that A is K-nonnegative and $(\mathcal{E}, \mathcal{P}(A,K))$ is given by Figure 1. Let $\tilde{K} = \mathrm{pos}\{(1-\alpha)x_1 + \alpha x_m, x_1, x_2, \ldots, x_{m-1}\}$. It is not difficult to show that for $\alpha > 1$, sufficiently close to 1, $(1-\alpha)x_1 + \alpha x_m, x_1, x_2, \ldots, x_{m-1}$ are precisely all the extreme vectors of $\tilde{K}$. Furthermore, A is $\tilde{K}$-nonnegative and $(\mathcal{E}, \mathcal{P}(A, \tilde{K}))$ is isomorphic to Figure 2 (under the isomorphism given by: $\Phi((1-\alpha)x_1 + \alpha x_m) \mapsto$

$\Phi(x_1), \Phi(x_j) \mapsto \Phi(x_{j+1})$ for $j = 1\ldots, m-1$).

Now it should be clear that Lemma 7.2 is still true if Figure 1 is replaced by Figure 2.

Below are two other questions on this topic that one may explore:

Question 9.4. If K is an n-dimensional minimal cone such that the relation for its extreme vectors has p vectors on one side and q vectors on the other side, where $p, q \geq 2, p+q \leq n+1$, what is $\gamma(K)$?

Question 9.5. Given positive integers m, n with $m \geq n$, determine

$$\min\{\gamma(K) : K \in \mathcal{P}(m,n)\}.$$

We suspect that for the polyhedral cone $K_0 \in \mathcal{P}(m,3)$ with extreme vectors $x_1, \ldots, x_m$ given by $x_j = (\cos\frac{2j\pi}{m}, \sin\frac{2j\pi}{m}, 1)^T$ for $j = 1, \ldots, m$, we have

$$\gamma(K_0) = \min\{\gamma(K) : K \in \mathcal{P}(m,3)\}.$$

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